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The Graph's Minimum Monopoly Energy

Ahmed Naji[1] * and N.D.Sonernam[1]

1 Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru, India.

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Abstract: A subset M ⊆ V (G) of a graph G(V,E) is referred to be a monopoly set of G if each vertex v ∈ V − M has at least one neighbor in M. The smallest cardinality of a monopoly set among all monopoly sets in G is its monopoly size, or mo(G). In this study, we determine the minimal monopoly energies of various typical graphs and introduce the minimum monopoly energy, or EM(G), of a graph G. We define upper and lower limits for EM(G).

MSC: 05C50, 05C99.

Keywords: Monopoly Set, Monopoly Size, Minimum Monopoly Matrix, Minimum Monopoly Eigenvalues, Minimum Monopoly Energy of a Graph.

1. Introduction

The term "graph G(V,E)" in this work refers to a simple graph, which is nonempty, finite, and devoid of loops, multiple edges, and directed edges. Let n and m represent G's vertex and edge counts, respectively. The number of vertices next to a vertex v in a graph G is its degree, which is represented by d(v). The set N(v) = {u ∈ V : uv ∈ E(G)} is the open neighborhood of any vertex v in a graph G. The degree of a vertex v ∈ V (G) with regard to a subset S is dS(v) = |N(v) ∩ S| for a subset S ⊆ V (G). The Harary book is consulted for graph theoretic nomenclature [9]. The monopoly size of a graph G, represented as mo(G), is a minimal cardinality of a monopoly set in G. A subset M ⊆ G is referred to be a monopoly set if each vertex v ∈ V −M has at least a neighbor in M. Specifically, monopolies are dynamic monopolies (dynamos) that, under an irreversible majority conversion procedure, will turn the whole graph black in the subsequent time step if it is colored black in a given time step. Peleg was the first to introduce dynamicos [15]. We consult [4-6, 13] for further information on dynamos in graphs. The author of [10] developed a monopoly set of a graph G, shown that every graph G admits a monopoly with at most) vertices, and proved that the mo(G) for a generic graph is at least. He also examined the connection between matchings and monopolies. I. Gutman [7] first proposed the idea of graph energy in 1978. Let A(G) = (aij) be the adjacency matrix of a graph G, which has n vertices and m edges. The eigenvalues of the graph G are the eigenvalues λ1, λ2,...,λn of a matrix A(G), considered to be in nonincreasing order. The spectrum of G is the multiset of eigenvalues of A(G), expressed as follows: Let λ1,λ2,...,λt for t ≤ n be the distinct eigenvalues of G with multiplicity m1,m2,...,mt, respectively.

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_t \\ m_1 & m_2 & \ldots & m_t \end{pmatrix}.$$

As *A* is real symmetric, the eigenvalues of *G* are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e. $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For more details on the mathematical aspects of the theory of graph energy we refer to [2, 8, 12]. Recently C. Adiga et al. [1] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover C. Motivated by this paper, we introduce minimum monopoly energy, denoted by $E_M(G)$, of a graph G, and computed minimum monopoly energies of some standard graphs. Upper and lower bounds for $E_M(G)$ are established. It is possible that the minimum monopoly energy that we are considering in this paper may be have some applications in chemistry as well as in other areas.

2. The Minimum Monopoly Energy of Graphs

Let G be a graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. any monopoly set M in a graph G with minimum cardinality is called a minimum monopoly set. Let M be a minimum monopoly set of G. The minimum monopoly matrix of G is the $n \times n$ matrix, denoted $A_M(G) = (a_{ij})$, where

$$aij = \begin{cases} 1, & \text{if } v_iv_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in M; \\ 0, & \text{otewise.} \end{cases}$$

The characteristic polynomial of $A_M(G)$, denoted by $f_n(G,\lambda)$, is defined as $f_n(G,\lambda) = det(\lambda I - A_M(G))$. The monopoly eigenvalues of G are the eigenvalues of $A_M(G)$. Since $A_M(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The minimum monopoly energy of G is defined as $E_M(G) = \sum_{i=1}^{n} |\lambda_i|$.

We first compute the minimum monopoly energy of a graph G in Fig. 1, to illustrious this concept

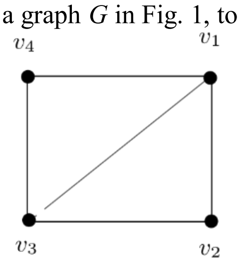

Figure 1 : *A graph G*

Example 2.1. *Let G be a graph in Figure 1, with vertices* v_1, v_2, v_3, v_4 *and let its minimum monopoly set be* $M_1 = \{v_1, v_3\}$. *Then minimum monopoly matrix of G is*

$$A_{M1}(G) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{M1}(G)$ *is* $f_n(G,\lambda) = \lambda^4 - 2\lambda^3 - 4\lambda^2$. *Then the minimum monopoly eigenvalues are* $\lambda_1 = 3.2361$, $\lambda_2 = \lambda_3 = 0$, $\lambda_4 = -1.2361$. *Therefore, the minimum monopoly energy of G is* $E_{M1}(G) = 4.4722$. *If we take another minimum monopoly set in G, namely* $M_2 = \{v_1, v_4\}$, *then*

1 1 1 1

$$A_{M2}(G) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial of $A_{M2}(G)$ is $f_n(G,\lambda) = \lambda^4 - 2\lambda^3 - 4\lambda^2 + \lambda + 1$. The minimum monopoly eigenvalues are $\lambda_1 = 3.1401$, $\lambda_2 = 0.57117$, $\lambda_3 = -0.43783$, $\lambda_4 = -1.2735$. Therefore, the minimum monopoly energy of G is $E_{M2}(G) = 5.4226$.

The examples above illustrate that the minimum monopoly energy of a graph G depends on the choice of the minimum monopoly set. i.e. the minimum monopoly energy is not a graph invariant.

3. Some Properties of Minimum Monopoly Energy of Graphs

In this section, we introduce some properties of characteristic polynomials of minimum monopoly matrix of a graph G and some properties of minimum monopoly eigenvalues.

Theorem 3.1. *Let G be a graph of order n, size m and monopoly size $mo(G)$. Let $f_n(G,\lambda) = c_0\lambda^n+c_1\lambda^{n-1}+c_2\lambda^{n-2}+\ldots+c_n$ be the characteristic polynomial of minimum monopoly matrix of G. Then*

1. $c_0 = 1$.
2. $c_1 = -mo(G)$.
3. $c_2 = \binom{mo(G)}{2} - m$.

Proof.

1. From the definition of $f_n(G,\lambda)$.
2. Since the sum of diagonal elements of $A_M(G)$ is equal to $|M| = mo(G)$, where M is a minimum monopoly set in G. The sum of determinants of all 1×1 principal submatrices of $A_M(G)$ is the trace of $A_M(G)$, which evidently is equal to $mo(G)$. Thus, $(-1)^1c_1 = mo(G)$.
3. $(-1)^2c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of $A_M(G)$, that is

$$
\begin{aligned}
c_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\
&= \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\
&= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ij}^2 \\
&= \begin{pmatrix} mo(G) \\ 2 \end{pmatrix} - m.
\end{aligned}
$$

□

Theorem 3.2. *Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of $A_M(G)$. Then*

(i) $\sum_{i}^{n} \lambda_i = mo(G)$.

(ii) $\sum_{i}^{n} \lambda_i^2 = mo(G) + 2m$.

Proof.

(i) Since the sum of the eigenvalues of $A_M(G)$ is the trace of $A_M(G)$, it follows that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |M| = mo(G)$$

.

(ii) Similarly the sum of squares of the eigenvalues of $A_M(G)$ is the trace of $(A_M(G))^2$. Then

$$
\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\
&= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i\neq j}^{n} a_{ij}a_{ji} \\
&= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j} a_{ij}^2 \\
&= mo(G) + 2m.
\end{aligned}
$$

□

Theorem 3.3. *Let G be a graph of order n and size m and let $\lambda_1(G)$ be the largest eigenvalue of $A_M(G)$. Then*

$$\lambda_1(G) \ge \frac{2m + mo(G)}{n}.$$

Proof. Let G be a graph of order n and let λ_1 be the largest minimum monopoly eigenvalue of $A_M(G)$. Then from [2] we have $\lambda_1 = \max_{X\neq 0}\left\{\frac{X^tAX}{X^tX}\right\}$, where X is any nonzero vector and X^t is its transpose and A is a matrix . If we tack

$X = J = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. Then we have $\lambda_1 \geq \frac{J^tA_M(G)J}{J^tJ} = \frac{2m+mo(G)}{n}$. □

Theorem 3.4. *Let G be a graph with a minimum monopoly set M. If the minimum monopoly energy $E_M(G)$ of G is a rational number, then $E_M(G) \equiv mo(G)\ (mod 2)$.*

Proof. Let $\lambda_1,\lambda_2,...,\lambda_n$ be the minimum monopoly eigenvalues of G of which $\lambda_1,\lambda_2,...,\lambda_r$ are positive and the rest are non-positive, then

$$\sum_{i=1}^{n}|\lambda_i| = (\lambda_1+\lambda_2+...+\lambda_r) - (\lambda_r+1+...+\lambda_n)$$

$$= 2(\lambda_1+\lambda_2+...+\lambda_r) - (\lambda_1+\lambda_2+...+\lambda_n).$$

Hence, By Theorem 3.2 we have $E_M(G) = 2(\lambda_1+\lambda_2+...+\lambda_r) - mo(G)$. Since $\lambda_1,\lambda_2,...,\lambda_r$ are algebraic integers, so is their sum. Therefore, $(\lambda_1+\lambda_2+...+\lambda_r)$ must be integer if $E_M(G)$ is rational. Hence, the Theorem. □

4. Minimum Monopoly Energy of Some Standard Graphs

In this section, we investigate the exact values of the minimum monopoly energy of some standard graphs. **Theorem 4.1.** *For the complete graph K_n, $n \geq 2$, the minimum monopoly energy is*

$$E_M(K_n) = \begin{cases} \frac{n-1}{2} + \sqrt{n^2-1}, & \text{if } n \text{ is odd;} \\ \frac{n-2}{2} + \sqrt{n^2-1}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Then the minimum monopoly size is

$$mo(K_n) = \lfloor\frac{n}{2}\rfloor = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd;} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Hence, the minimum monopoly set either $\{v_1, v_2, \cdots, v_{\frac{n-1}{2}}\}$ if n is odd or $\{v_1, v_2, \cdots, v_{\frac{n}{2}}\}$ if n is even. Then, we consider the following two cases:

Case 1: If n is odd,

$$A_M(K_n) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}_{n\times n}$$

The respective characteristic polynomial is

$$\begin{vmatrix} 1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & \lambda & -1 & \cdots & -1 & -1 \\ 1 & -1 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & -1 & \cdots & \lambda & -1 \\ 1 & -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{n\times n}$$

$$f_n(K_n,\lambda) = \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \\ \vdots & \vdots \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ \vdots & \vdots \\ -1 & -1 \\ -1 & -1 \end{vmatrix}$$

$$= \lambda^{\frac{n-3}{2}}(\lambda+1)^{\frac{n-1}{2}}\left(\lambda^2-(n-1)\lambda-\frac{n-1}{2}\right).$$

The minimum monopoly spectrum of K_n will be written as

$$MM\ Spec(K_n) = \begin{pmatrix} 0 & -1 & \frac{(n-1)+\sqrt{n^2-1}}{2} & \frac{(n-1)-\sqrt{n^2-1}}{2} \\ \frac{n-3}{2} & \frac{n-1}{2} & 1 & 1 \end{pmatrix}.$$

Hence, the minimum monopoly energy of a complete graph in this case is

$$E_M(K_n) = \frac{n-1}{2} + \sqrt{n^2-1}.$$

Case 2: If n is even,

$A_M(K_n) =$

1 1 1 ⋯ 1 1 ⋯ 1 1

1 1 1 ⋯ 1 1 ⋯ 1 1

1 1 1 ⋯ 1 1 ⋯ 1 1

. . . ⋯ . . . .

1 1 1 ⋯ 1 0 ⋯ 1 1

. . . ⋯ . . . .

1 1 1 ⋯ 1 1 ⋯ 1 1

1 1 1 ⋯ 1 1 ⋯ 0 1

1 1 1 ⋯ 1 0

1 1 1 ⋯

$n\times n$

The respective characteristic polynomial is

$$f_n(K_n,\lambda)=\begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda-1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{n\times n}$$

$$=\lambda^{\frac{n-2}{2}}(\lambda+1)^{\frac{n-2}{2}}(\lambda^2-(n-1)\lambda-\frac{n}{2}).$$

The minimum monopoly spectrum of K_n is

$$MM\ Spec(K_n)=\begin{pmatrix} 0 & -1 & \frac{(n-1)+\sqrt{n^2+1}}{2} & \frac{(n-1)-\sqrt{n^2+1}}{2} \\ \frac{n-2}{2} & \frac{n-2}{2} & 1 & 1 \end{pmatrix}.$$

Hence, the minimum monopoly energy of a complete graph in this case is $E_M(K_n)=\frac{n-2}{2}+\sqrt{n^2+1}$. □

Theorem 4.2. *For the complete bipartite graph $K_{r,s}$, for $r\le s$, the minimum monopoly energy is equal to* $(r-1)+\sqrt{4rs+1}$.

Proof. For the complete bipartite graph $K_{r,s}$, for $r\le s$ with vertex set $V=\{v_1,v_2,\cdots,v_r,v_1,v_2,\cdots,v_s\}$. The minimum monopoly set is $M=\{v_1,v_2,\cdots,v_r\}$.

Then

$$A_M(K_{r,s}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & & & \ddots & \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots & & & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(r+s)\times(r+s)}$$

The characteristic polynomial of $A_M(K_{r,s})$ is

$$f_n(K_{r,s},\lambda)=\begin{vmatrix}\lambda-1 & 0 & 0 & 0 & -1 & -1 & -1 & \cdots & -1\\ 0 & \lambda-1 & 0 & 0 & -1 & -1 & -1 & \cdots & -1\\ 0 & 0 & \lambda-1 & 0 & -1 & -1 & -1 & \cdots & -1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \lambda-1 & -1 & -1 & -1 & \cdots & -1\\ -1 & & -1 & 1 & \lambda & 0 & 0 & \cdots & 0\\ -1 & & -1 & 1 & 0 & \lambda & 0 & \cdots & 0\\ -1 & & -1 & 1 & 0 & 0 & \lambda & \cdots & 0\\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & & -1 & 1 & 0 & 0 & 0 & \cdots & \lambda\end{vmatrix}_{(r+s)\times(r+s)}$$

$\cdots$

$\cdots$

$\cdots$

...

$\cdots$

-1 $\cdots$ $-$

-1 $\cdots$ $-$

-1 $\cdots$ $-$

. ...

.

.

-1 $\cdots$ $-$

$$=\lambda^{r-1}(\lambda-1)^{s-1}(\lambda^2-\lambda-rs).$$

and

$$MM\ Spec(K_{r,s})=\begin{pmatrix}0 & -1 & \frac{1+\sqrt{4rs+1}}{2} & \frac{1-\sqrt{4rs+1}}{2}\\ r-1 & s-1 & 1 & 1\end{pmatrix}$$

Hence, $E_M(K_{r,s})=(r-1)+\sqrt{4rs+1}$. □

Theorem 4.3. *For a star graph $K_{1,n-1}$, $n\geq 2$, the minimum monopoly energy is equal to* $\sqrt{4n-3}$.

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V=\{v_0,v_1,v_2,\cdots,v_n\}$, where v_0 is the center vertex. The minimum monopoly set of $k_{1,n-1}$ is $M=\{v_0\}$. Then

$$A_M(K_{1,n-1})=\begin{pmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & 0 & 0 & \cdots & 0\\ 1 & 0 & 0 & \cdots & 0\end{pmatrix}$$

. . . .

... . . .

.

. . . 0

1 0 0 ⋯

$n\times n$

The characteristic polynomial of $A_M(K_{1,n-1})$ is

$$f_n(K_{n,m}),\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \lambda^{n-2}(\lambda^2-\lambda-(n-1)).$$

and

$$MM\ Spec(K_{1,n-1}) = \begin{pmatrix} 0 & \frac{1+\sqrt{4n-3}}{2} & \frac{1-\sqrt{4n-3}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Therefore, the minimum monopoly energy of a star graph is $E_M(K_{1,n-1}) = \sqrt{4n-3}$. □

Definition 4.4. *The double star graph $S_{n,m}$ is the graph constructed from union $K_{1,n-1}$ and $K_{1,m-1}$ by join whose centers v_0 with u_0. Then $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0,v_1,...,v_{n-1},u_0,u_1,...,u_{m-1}\}$ and edge set $E(S_{n,m}) =$ $\{v_0u_0, v_0v_i, u_0u_j | \ 1 \le i \le n-1, 1 \le j \le m-1\}$.*

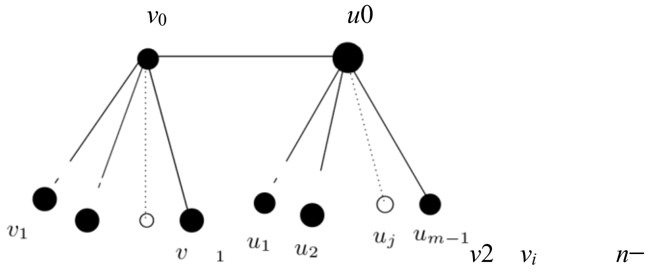

Figure 2 : Double Star Graph $S_{n,m}$

Theorem 4.5. *For the double star graph $S_{t,t}$, with $t \ge 3$, the minimum monopoly energy is equal to $2(\sqrt{t-1}+\sqrt{t})$.*

Proof. For the double star graph $S_{t,t}$ with $V = \{v_0,v_1,...,v_t-1,u_0,u_1,...,u_t-1\}$ the minimum monopoly set is $M = \{v_0,u_0\}$. Then

$$A_M(S_{t,t}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}_{2t\times 2t}$$

The characteristic polynomial of $A_M(S_{t,t})$ is

$$f_n(S_{t,t}), \lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & \lambda-1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & \lambda \end{vmatrix}_{2t\times 2t}$$

$$= \lambda^{2t-4}(\lambda^2 - (t-1))(\lambda^2 - 2\lambda - (t-1)).$$

Then the minimum monopoly spectrum of $S_{t,t}$ is

$$MM\ Spec(S_{t,t}) = \begin{pmatrix} 0 & \sqrt{t-1} & -\sqrt{t-1} & 1+\sqrt{t} & 1-\sqrt{t} \\ 2t-4 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Hence, the minimum monopoly energy of $S_{t,t}$ is

$$E_M(S_{t,t}) = 2(\sqrt{t-1} + \sqrt{t}).$$

□

Definition 4.6. *The crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1,u_2,...,u_n,v_1,v_2,...,v_n\}$ and edge set $\{u_iv_i : 1 \leq i,j \leq n,\ i =6\ j\}$. Therefore S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Theorem 4.7. *For $n \geq 3$, the minimum monopoly energy of the crown graph S_n^0 is equal to* $\sqrt{5}(n-1) + \sqrt{4n^2 - 8n + 5}$.

Proof. For the crown graph S_n^0 with vertex set $V = \{u_1,u_2,...,u_n,v_1,v_2,...,v_n\}$, the subset $M = \{u_1,u_2,...,u_n\}$ is a minimum monopoly set in S_n^0. Then

$$A_M(S_n^0) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n \times 2n}.$$

The Characteristic polynomial of $A_M(S_n^0)$ is

$$f_n(S_n^0,\lambda) = \begin{vmatrix} \lambda-1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \lambda-1 & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\ 0 & 0 & \lambda-1 & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda-1 & -1 & -1 & -1 & \cdots & 0 \\ 0 & & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\ -1 & & & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}$$

−1

−1 0

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−1

$= (\lambda^2 - \lambda - 1)^{n-1}(\lambda^2 - \lambda - (n-1)^2)$.

Then, the minimum monopoly spectrum of S_n^0 is

$$MM\ Spec(S_n^0) = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{4n^2-8n+5}}{2} & \frac{1+\sqrt{4n^2-8n+5}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Therefore, the minimum monopoly energy of S_n^0 is $E_M(S_n^0) = (n-1)\sqrt{5} + \sqrt{4n^2 - 8n + 5}$. □

5. Bounds on Minimum Monopoly Energy of Graphs

Theorem 5.1. *Let G be a connected graph of order n and size m. Then*

$$\sqrt{2m + mo(G)} \le E_M(G) \le \sqrt{n(2m + mo(G))}$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$ and by Theorem 3.2, we get

$$(E_M(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \lambda_i^2\right) \le n\,(2m + mo(G)).$$

Therefore, the upper bound is hold. Now, since $\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \ge \sum_{i=1}^{n} \lambda_i^2$, it follows by Theorem 3.2 that $(E_M(G))^2 \ge 2m + mo(G)$. Therefore, the lower bound is hold. □

Theorem 5.2. *For a connected graph G of order n and size m.*

$$\sqrt{n+1} \le E_M(G) \le n\sqrt{n}.$$

Proof. Since for any graph $mo(G) \leq \frac{n}{2}$ (see [10]), it follows that by using Theorem 5.1 and well-known result $2m < n^2 - n$, we have

$$E_M(G) \leq \sqrt{n\,(2m + mo(G))} \leq \sqrt{n\left[(n^2 - n) + \frac{n}{2}\right]} \leq n\sqrt{n}.$$

For the lower bound, Since for any connected graph $n \leq 2m$ and $mo(G) \geq 1$ (see [14]), it follows by Theorem 5.1 that

$$E_M(G) \geq \sqrt{2m + mo(G)} \geq \sqrt{n + 1}.$$

□

Similar to Koolen and Moultons [11], upper bound for $E_M(G)$ is given in the following theorem.

Theorem 5.3. *Let G be a graph of order n and size m. Then*

$$E_M(G) \leq \frac{2m + mo(G)}{n} + \sqrt{(n-1)\left[2m + mo(G) - (\frac{2m + mo(G)}{n})^2\right]}.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2).$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$, we have

$$\left(\sum_{i=2}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=2}^{n} 1\right)\left(\sum_{i=2}^{n} \lambda_i^2\right).$$

Hence, by Theorem 3.2 we have

$$(E_M(G) - |\lambda_1|)^2 \leq (n-1)(2m + mo(G) - \lambda_1^2).$$

Therefore,

$$E_M(G) \leq \lambda_1 + \sqrt{(n-1)(2m + mo(G) - \lambda_1^2)}.$$

From Theorem 3.3 we have $\lambda_1 \geq \frac{2m+mo(G)}{n}$.

Since $f(x) = x + \sqrt{(n-1)(2m + mo(G) - x^2)}$ is a decreasing function, we have

$$f(\lambda_1) \leq f(\frac{2m + mo(G)}{n}).$$

Thus,

$$E_M(G) \leq f(\lambda_1) \leq f(\frac{2m + mo(G)}{n}).$$

Therefore,

$$E_M(G) \leq \frac{2m + mo(G)}{n} + \sqrt{(n-1)\left[2m + mo(G) - (\frac{2m + mo(G)}{n})^2\right]}.$$

□

Theorem 5.4. *Let G be a connected graph of order n and size m. If $D = det(A_M(G))$, then*

$$E_M(G) \geq \sqrt{2m + mo(G) = n(n-1)D^{2/n}}.$$

Proof. Since

$$(E_M(G))^2 = \Big(\sum_{i=1}^{n}|\lambda_i|\Big)^2 = \Big(\sum_{i=1}^{n}|\lambda_i|\Big)\Big(\sum_{i=1}^{n}|\lambda_i|\Big) = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{i<j}|\lambda_i||\lambda_j|.$$

Using the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \Big(\prod_{i\neq j}|\lambda_i||\lambda_j|\Big)^{1/[n(n-1)]}.$$

Hence, by this and Theorem 3.2 we get

$$\begin{aligned}(E_M(G))^2 &\geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\Big(\prod_{i\neq j}|\lambda_i||\lambda_j|\Big)^{1/[n(n-1)]}\\ &\geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\Big(\prod_{i=j}|\lambda_i|^{2(n-1)}\Big)^{1/[n(n-1)]}\\ &= \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\Big|\prod_{i=j}\lambda_i\Big|^{2/n}\\ &= 2m + mo(G) + n(n-1)D^{2/n}.\end{aligned}$$

□

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