International Journal of Mathematical Applications and Solutions
Vol 1, Issue 2 2025

The resolution of a viscoelastic-related partial integro-differential problem

Dharani.K, sree Kumari

Department of Applied Mathematics,

Mathlogix publications

Article Info

Received: 28-03-2025 Revised:04-04-2025 Accepted:15-04-2025 Published:26-04-2025

Abstract

A wide variety of scientific and engineering domains make use of partial integro-differential equations. Several numerical methods for solving a partial integro-differential equation with a weakly sin-gular kernel are presented in this work. A partial integro-differential equation with an initial condition and specified boundary conditions may be solved using these approaches. Based on finite-difference methods and the product trapezoidal numerical integration rule, we design many computational algorithms. The modeling of physical processes incorporating viscoelasticity forces presents this challenge. The implicit approach of Crandall, the Crank-Nicolson formula, the forward-Euler explicit scheme, the backward-Euler implicit technique, and the finite-difference methods established are all based on these frameworks. In terms of the space variable, three of the approaches exhibit accuracy at the second order. To a greater extent than the alternatives, Crandall's plan is accurate. To ensure the consistency and precision of the methods described, the numerical solutions to a test issue are provided.

Keywords: Partial integro-differential equation; Finite-difference schemes; Order of accuracy; Viscoelasticity; Product numerical integration rule

C.R. Category: G.1.9

1. Introduction

In the last few years, it has become increasingly apparent that many physical phenomena can be described by partial integro-differential equations with weakly singular kernels. For example, these equations occur in the study of many viscoelastic phenomena [1]. In recent years there has been increasing attention in the literature to the development, analysis and implementation of accurate methods for the numerical solution of non-linear partial-integro differential equations [2]. In this paper we present an investigation of the use of finite-difference schemes and numerical integration procedures for the numerical solution of the following problem:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \int_0^t (t-s)^{-1/2} u_{xx}(x, s)\mathrm{d}s, \tag{1}$$

with the initial condition

$$u(x, 0) = f(x), \quad 0 \le x \le 1, \tag{2}$$

and the boundary conditions

$$u(0,t) = g_1(t), \quad 0 \le t \le T, \tag{3}$$

$u(1,t) = g_2(t), \quad 0 \le t \le T,$ (4)

where f, g_1 and g_2 are known functions, and the function u is unknown. This problem occurs in the modelling of physical phenomena involving viscoelastic forces [3, 4].

Numerical investigations of this and some other similar problems have been performed by several authors [1, 5–9], but most of them considered smooth integral kernels only. The method presented in [10] treats the weakly singular integral kernel (1) by means of the product trapezoidal numerical integration technique. The convergence properties of the numerical schemes presented are investigated in [11].

Lopez-Marcos [5] established the existence and uniqueness of the solutions to this problem under the following assumptions.

(1) u_{tt}, u_{ttt} and u_{xxtt} are continuous for $0 \le x \le 1$ and $0 < t \le T$.
(2) There exists a positive constant C such that, for $0 \le x \le 1$ and $0 < t \le T$, the following hold:

$$|u_{tt}(x, t)| \le Ct^{-1/2}, \quad (5)$$

$$|u_{ttt}(x, t)| \le Ct^{-3/2}, \quad (6)$$

$$|u_{xxtt}(x, t)| \le Ct^{-1/2}. \quad (7)$$

This and similar problems [12–14] can also be found in the modelling of heat flow in materials with memory [15, 16], and in linear viscoelastic mechanics [17, 18]. In linear viscoelastic problems, the integral term (1) represents the viscosity part of the equation.

The aim of this work was to develop computational algorithms to obtain the solution of problem (1)–(4).

Finite-difference methods are widely used to solve differential and partial differential equations. A large number of researchers have employed this scheme to solve boundary value problems. The advantage of finite-difference formulae is their simplicity. For instance, they can handle various boundary conditions and non-uniform meshes quite easily. In applications such as heat transfer at a boundary, the boundary conditions can range from Dirichlet (perfect conducting) to Neumann (perfect insulating) to the more general Robin type. Finite-difference formulae can treat all these boundary conditions equally well.

This paper is organized as follows. Some notation is introduced in section 2, which also includes discussion of the treatment of the singular kernel and the explicit and implicit finite-difference formulae. Two implicit two-level three-point techniques and an explicit two-level three-point procedure are discussed. We perform a numerical test for the finite-difference discretization presented and report the results in section 3. Finally we conclude this paper with a brief discussion.

2. The finite-difference techniques

We introduce a grid $x_i = ih, j = 0, 1, \ldots, M,$ where $h = 1/M$ and M is a positive integer. The step length in time is denoted by k, $k = T/N$ where N is a positive integer, and the subscript n refers to the time level $t_n = nk, n = 0, 1, \ldots, N$.

The finite-difference method is a classical approach to the solution of linear partial differential equations. It is an approximate method, given that the partial derivatives at a point are approximated by a difference ratio over a small interval. Depending on the expression of this difference ratio, there are various versions of the model, including the explicit, implicit and the Crank–Nicolson techniques. There is no generally proved theory for the non-linear case, which is the equation for our model. The procedure described below depends on the type of non-linearity. There are occasions when linearization of the problem is recommended. However, there are also cases when it is possible to generalize the methods indicated previously [19].

The finite-difference formulae described in this section which will be applied at interior grid points in the solution domain are the three-point second-order explicit finite difference technique and the three-point second-order implicit scheme [20], the Crank–Nicolson formula [11] and Crandall's fourth-order implicit technique [2] which approximate the solution of our non-linear partial integro-differential equation.

Using the initial condition $u(x, 0) = f(x)$, $0 \le x \le 1$, we solve equation (1) approximately at spatial points x_i, commencing with initial values $y^0 = f(x_i)$, $i = 0, 1, 2, \ldots, M$, and boundary values

$$\frac{u_i^{n+1} - u_i^n}{k} \simeq -(u_i^n)\,\frac{(u_{i+1}^n - u_{i-1}^n)}{2h} + \int_0^{t_n} (t_n - s)^{-1/2}\quad \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}\ \mathrm{d}s. \tag{8}$$

It can easily be seen that using the product trapezoidal numerical integration rule [12] to approximate $\int_0^t (t - s)^{-1/2} f(s)\mathrm{d}s$ numerically gives

$$\begin{aligned}\int_0^{t_n} (t_n - s)^{-1/2} f(s)\mathrm{d}s &= \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \vartheta^{-1/2} f(t_n - \vartheta)\mathrm{d}\vartheta \\ &\simeq \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \vartheta^{-1/2}\quad \frac{t_{j+1} - \vartheta}{k} f(t_{n-j}) + \frac{\vartheta - t_j}{k} f(t_{n-j-1})\ \mathrm{d}\vartheta \\ &\simeq \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \vartheta^{-1/2} f(t_n - \vartheta)\,\mathrm{d}\vartheta \simeq A_n f(t_0) + \sum_{p=0} c_p f(t_{n-p}),\end{aligned} \tag{9}$$

where

$$A_n = 2\ t_n^{1/2} - \frac{1}{k} \int_{t_p}^{t_{n+1}} \vartheta^{1/2}\mathrm{d}\vartheta\quad ,\quad n \ge 0, \tag{10}$$

$$c_0 = \frac{2}{k} \int_0^{t_1} \vartheta^{1/2}\,\mathrm{d}\vartheta, \tag{11}$$

$$c_p = \frac{2}{k} \int_{t_p}^{t_{p+1}} \vartheta^{1/2}\,\mathrm{d}\vartheta - \int_{t_{p-1}}^{t_p} \vartheta^{1/2}\mathrm{d}\vartheta\quad ,\quad p \ge 1. \tag{12}$$

It can easily be seen that

$$A_n = 2t_n^{1/2} - \frac{4}{3k} t^{3/2}_{n+1} + \frac{4}{3k} t^{3/2}_{n}, \tag{13}$$

$$c_0 = \frac{4}{3k}(t_1^{3/2} - t_0^{3/2}), \tag{14}$$

$$c_p = \frac{4}{3k} t_{p+1}^{3/2} + t_{p-1}^{3/2}\ . \tag{15}$$

It should be noted that we have used the following in (9):

$$f(t_n - \vartheta) \simeq \frac{t_{j+1} - \vartheta}{k} f(t_{n-j}) + \frac{\vartheta - t_j}{k} f(t_{n-j-1}), \tag{16}$$

for $\vartheta \in [t_j, t_{j+1}]$ with $0 \le j \le n-2$.

Using equation (9) for the Crank–Nicolson method for (1) gives [21]

$$\frac{u_i^{n+1} - u_i^n}{k} \simeq -\frac{u_i^{n+1} + u_i^n}{2} \frac{u_{i+1}^n - u_{i-1}^n}{2h} + D_n(u^0_{i-1} - 2u_i^0 + u^0_{i-1}) + \sum_{p=0}^{n} c_p \frac{u_{i-1}^{n-p} - 2u_i^{n-p} + u_{i+1}^{n-p} + u_{i-1}^{n+1-p} - 2u_i^{n+1-p} + u_{i+1}^{n+1-p}}{2h^2}, \tag{17}$$

where $i = 1, 2, \ldots, M-1$ and

$$D_n = \frac{A_n + A_{n+1} + c_{n+1}}{2h^2}, \quad n \ge 0. \tag{18}$$

Note that using equation (9) with the backward Euler method for (1) gives

$$\frac{u_i^{n+1} - u_i^n}{k} \simeq -u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2h} + D_n(u^0_{i-1} - 2u_i^0 + u^0_{i-1}) + \sum_{p=0}^{n} c_p \frac{u_{i-1}^{n+1-p} - 2u^{n+1-p} + u_{i+1}^{n+1-p}}{h^2}. \tag{19}$$

If we use the forward Euler method [5] for (1), we obtain

$$\frac{u_i^{n+1} - u_i^n}{k} \simeq -(u_i^n) \frac{u_{i+1}^n - u_{i-1}^n}{2h} + D_n(u^0_{i-1} - 2u_i^0 + u^0_{i-1}) + \sum_{p=0}^{n} c_p \frac{u_{i-1}^{n-p} - 2u^{n-p} + u_{i+1}^{n-p}}{h^2}, \tag{20}$$

where $1 \le i \le M-1, \quad 0 \le n \le N-1$.

For the numerical solution of (17) with initial condition

$$u_i^0 = f(x_i), \quad 1 \le i \le M-1, \tag{21}$$

and boundary conditions

$$u_0^n = 0, \quad 0 \le n \le N, \tag{22}$$

$$u_M^n = 0, \quad 0 \le n \le N, \tag{23}$$

we obtain a linear algebraic system in the form

$$P u^{n+1} = b^n, \quad 0 \le n \le M-1, \tag{24}$$

where P is a symmetric tridiagonal matrix $(M-1) \times (M-1)$. It can easily be seen that the diagonal elements of P for (17) are in the following form:

$$P_{i,i} = 1 + \frac{kc_0}{h^2}, \tag{25}$$

where the subdiagonal elements are $-kc_0/2h^2$.

The convergence analysis for the Crank–Nicolson scheme is presented in [8].

Using the backward Euler implicit scheme gives the following equation:

$$\frac{u_i^{n+1} - u_i^n}{k} \simeq -(u_i^{n+1}) \quad \frac{u_{i+1}^n - u_{i-1}^n}{2h} + D_n(u^0{}_{i-1} - 2u_i^0 + u_{i-1}^0) + \sum_{p=0}^{n} c_p \frac{u_{i-1}^{n+1-p} - 2u_i^{n+1-p} + u_{i+1}^{n+1-p}}{h^2}, \tag{26}$$

where $i = 1, 2, \ldots, M - 1$.

If we use the Crandall's procedure [22] for (1), we have

$$\frac{u_i^{n+1} - u_i^n}{k} \simeq - \quad \frac{u_i^{n+1} + u_i^n}{2} \quad \frac{u_{i+1}^n - u_{i-1}^n}{2h} + D_n(u^0{}_{i-1} - 2u_i^0 + u_{i-1}^0) + \sum_{p=0} c_p \quad \frac{1}{2} - \frac{1}{12s} \quad \frac{u_{i-1}^{n-p} - 2u_i^{n-p} + u_{i+1}^{n-p}}{2h^2} + \quad \frac{1}{2} + \frac{1}{12s} \quad \frac{u_{i-1}^{n+1-p} - 2u_i^{n+1-p} + u_{i+1}^{n+1-p}}{2h^2} \quad , \tag{27}$$

where $i = 1, 2, \ldots, M - 1$.

Note that D_n in (17), (19) and (20) can be obtained similarly to the way in which A_n is obtained in (13).

It can be seen that the kernel function in (1) has a weak singularity at the origin. This is particularly interesting in viscoelasticity, because it might smooth the solution when the boundary data are discontinuous [18].

The finite-difference schemes developed in this paper are very easy to implement for similar problems with Neumann boundary conditions. The methods developed are computationally attractive and applications are demonstrated using an illustrative example in the next section.

3. Numerical test

To illustrate the description above and to test the finite-difference schemes developed here for solving the partial integro-differential equation with a weakly singular kernel, we give a numerical example.

Consider (1)–(4) with

$$u_t = \int_0^t (t - s)^{-1/2} u_{xx}(x, s)\mathrm{d}s, \tag{28}$$

$$u(0,t) = 0, \quad 0 \le t \le T, \tag{29}$$

$$u(1,t) = 0, \quad 0 \le t \le T, \tag{30}$$

$$u(x, 0) = \sin(\pi x), \quad 0 \le x \le 1. \tag{31}$$

It can easily be verified that $\sum_{n=0}^{\infty} 2^{n+1}$

$$u(x, t) = (-1)^n \hat{W} \quad \frac{3}{1} \quad -$$

$$(\pi^{5/2}t^{3/2}{}^{n} \quad \sin \quad)$$
$$(\pi$$
$$x), \qquad (32)$$

Table 1. Results for u with $h = 1/50$, $s = 1/4$ and $T = 1.0$.

		Error			
x	Exact solution	Three-point explicit	Three-point implicit	Crandall's method	Crank–Nicolson
0.1	0.00796105	7.5×10^{-3}	7.1×10^{-3}	6.2×10^{-4}	5.1×10^{-3}
0.2	0.01514283	7.5×10^{-3}	7.2×10^{-3}	6.1×10^{-4}	5.2×10^{-3}
0.3	0.02084231	7.6×10^{-3}	7.4×10^{-3}	6.5×10^{-4}	5.3×10^{-3}
0.4	0.02450162	7.4×10^{-3}	7.5×10^{-3}	6.6×10^{-4}	5.2×10^{-3}
0.5	0.02576251	7.5×10^{-3}	7.5×10^{-3}	6.6×10^{-4}	5.4×10^{-3}
0.6	0.02450161	7.4×10^{-3}	7.3×10^{-3}	6.6×10^{-4}	5.3×10^{-3}
0.7	0.02084231	7.3×10^{-3}	7.2×10^{-3}	6.4×10^{-4}	5.5×10^{-3}
0.8	0.01514283	7.7×10^{-3}	7.4×10^{-3}	6.5×10^{-4}	5.3×10^{-3}
0.9	0.00796105	7.8×10^{-3}	7.6×10^{-3}	6.4×10^{-4}	5.2×10^{-3}

Given the given beginning condition and all boundary conditions, is the analytical solution that satisfies the partial integro-differential equation (1).
Table 1 displays the outcomes acquired after 104 time steps (k = 10−4) for the variable un at T = 1.0 computed for h = 0.02 using the three-point explicit finite-difference method, the three-point implicit technique, the Crank-Nicolson procedure, and the (3,3) Crandall finite-difference scheme. An important difference between the explicit and implicit Euler methods is the size of their maximum errors [11]. It should be kept in mind while comparing that the three-point implicit strategy can reduce Δx = h with a fixed Δt = k, but it doesn't produce the temporal instability that could be present in the explicit technique. As we go on to the Crank-Nicolson scheme, we see that it outperforms the three-point implicit method considerably without incurring any extra computational expense. In contrast, Crandall's algorithm produces much better outcomes with zero additional computing load. In addition, as shown in table 1, the Crandall approach continues to surpass all other strategies even when we cut the number of spatial gridpoints in half. Despite using a quarter as many gridpoints, it outperforms both the explicit and implicit systems and achieves accuracy that is on par with the Crank-Nicolson scheme.
Despite discussing Crandall's finite difference method, the results obtained for different choices of s did not represent its fourth-order convergence. For the Crank-Nicolson finite-difference formula, the whole technique fails to attain second-order convergence within the allotted time.
The disparity between the explicit and implicit techniques of Euler was hardly discernible at h = 0.04. When we further decreased the grid spacing h from 0.04 to 0.01, the two methods showed a maximum relative difference of 20%.

4. Discussion and conclusion

In order to compute the solutions of a partial integro-differential equation with a weakly singular kernel, which arises in the modeling of physical events involving viscoelastic forces, four numerical approaches have been examined. A method for studying a specific class of partial integro-differential equations has been described in this research. The partial integro-differential equation with a weakly singular kernel was solved using the three-point explicit finite-difference methodology, the three-point implicit formula, the Crank-Nicolson (3,3) scheme, and Crandall's implicit method. To get close to the integral term, the product trapezoidal method is used. These models were successfully solved using the numerical approach that was suggested. When applied to a real-world scenario, the novel algorithms described here demonstrated promising outcomes.

leading to the convergence of the precise answer as h approaches zero. Although the Dirichlet boundary issue was the original inspiration for the finite-difference method, it works just as well for other boundary problems, such as the Neumann and Robin problems.

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