The Dual-Matrix Approach to Transportation Problem Solving: Investigation

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Abstract: A transportation decision-making problem is considered. It is shown that the solution exists and that the solution space may be described. We take a look at the two sides of the transportation issue. A systematic approach is detailed, and a solution to the transportation problem is proposed using an algorithm. Each stage of the procedure is shown with a numerical example.

Keywords: Linear programming models, Transportation problem, Dual- Matrix, Dual simplex.

1. Introduction

The transportation problem is an example of a linear programming paradigm. A substantial amount of research has been conducted on the subject of transportation issues. The works of Arsham and A.B. Kahn [5], Balinski and R.E. Gomory [6], Charnes and W.W. Cooper [1], Ford and D.R. Fulkerson [7], Shafaat and S.K. Goyal [9], and Ping and K.F. Chu [2] are significant contributions to the field of transportation studies. To solve the transportation problem, Ping and CHU [2] developed the dual-matrix approach. By changing the original transportation problem into a matching dual transportation problem, they hope to find a better first fundamental feasible solution. Additionally, they mentioned that there may be space and time savings when going from the original simplex table to the optimal simplex table. Finding a solution and analytically describing the solution space are both made more difficult in their paper. In addition, there is no converging trend in the solution. This research extends the approach given in [2] by strengthening its mathematical foundations (existence, characterization, and convergence).

The Problem and its Dual

The transportation problem can be expressed as a linear programming model as follows:

$$\text{LP(1) Maximize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\begin{aligned} &\sum_{j=1}^{n} x_{ij} \le a_i \quad (i = i,...,m) \\ &\sum_{i=1}^{m} x_{ij} \ge b_i \quad (j = i,...,n) \\ &x_{ij} \ge 0 \ (i = 1,...,m;\ j = 1,dots,n) \end{aligned} \tag{1}$$

Here a_i and b_j are assumed to be positive, a_i and b_j are called supplies and demands. The cost c_{ij} are all nonnegative.

Dual of the L.P(1) is as follows:

LP(2) Maximize $\psi = \sum_{j=1}^{n} b_j v_j - \sum_{i=1}^{m} a_i u_i$

Subject to

$$v_j - u_i \le c_{ij} \quad (i = 1,...,m;\ j = 1,dots,n)$$

$$u_i, v_j \ge 0 \ (i = 1,...,m;\ j = 1,dots,n) \tag{2}$$

2. Mathematical Analysis of the Solution and Characterization of the Solution Space

Our approach to find the good solution is the dual approach. So from L.P(2) $u_i = 0, v_j = c_{rj} = \min_i c_{ij}$ gives the basic solution. ie., minimum cost variable is selected from every column of the problem. These variables are called basic variables.

In this way we have n basic variables corresponding to their respective cells.

Since L.P (1) has total supply available at each origin is greater than or equal to the required demand. ie., $\sum_{i=1}^{m} a_i \ge \sum_{j=1}^{n} b_j$,

we can create an additional destination D_0 with demand $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$, with shipping cost zero from each origin. Or we can say that, to make the problem balance we need to add m virtual cells having transporting cost zero. These cells are denoted as (1,0)(2,0)...(m,0) and treated as basic cells. Actually these cells do not exist in the original problem so called virtual

cells.

In this manner total (m+n) basic cells are (i_1,1)(i_2,2)...(i_n,n)(1,0)(2,0)...(m,0) and corresponding variables are x_{i1}, x_{i2},..., x_{ij}, x_{in}, x_{10}, x_{20},..., x_{m0} rest variables are non-basic variables.

Next L.P (1) is converted into standard L.P form then converted into matrix form. According to basic and non-basic data matrices and variables are partitioned into basic and non-basic respectively. To make the problem (1) in standard linear programming form by introducing only surplus variables we rewrite the problem as follows.

m n

Maximize $Z = \sum_{i=1}^{P}\sum_{j=1}^{P} c_{ij}x_{ij}$

Subject to

$$\sum_{i=1}^{m} xij = bj$$

$$-\sum_{j=1}^{m} xij - xi0 = -ai$$

$$xij, x0j \geq 0$$

$$Min\ Z = CX$$

$$AX = \begin{bmatrix} b \\ \\ -a \end{bmatrix} \quad (3)$$

$$X \geq 0$$

Partition the set of variables x_{ij} into x^{B}_{ij} and x^{NB}_{ij} and c_{ij} into c_{ij}^{B} and c^{NB}_{ij}. Where NB and B is defined as the set of indices for the non-basic and basic variables. A is partition in B and N matrices where B contains those columns of A which are associated with basic variables and N contains those columns which are associated with non-basic variables. Here B is square matrix of order (m+n) and N is of order $(m + n) \times mn$ then problem (3) can be written as

$$\begin{bmatrix} z \\ \\ xBij \\ \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -cBij & -cNBij \\ \\ 0 & B & N \end{bmatrix} \begin{bmatrix} 0 \\ \\ b \\ \\ -a \end{bmatrix} \quad (4)$$

Initially all non-basic variables are at zero level therefore the standard problem is reduced to

$$\begin{bmatrix} z \\ x_{ij}^{B} \end{bmatrix} = \begin{bmatrix} 1 & -c_{ij}^{B} \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 \\ b \\ -a \end{bmatrix}$$

If $T = \begin{bmatrix} 1 & -c_{ij}^{B} \\ 0 & B \end{bmatrix}$ then $(T)^{-1}$ is calculated using (A-2-7) page no.822 from Taha [4].

$$(T)^{-1} = \begin{bmatrix} 1 & cBijB \\ \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} z \\ \\ xij \\ \\ B \end{bmatrix} = \begin{bmatrix} c_{ij}^{B} B \begin{bmatrix} b \\ \\ -a \end{bmatrix} \\ \\ b \\ \\ B \end{bmatrix}$$

$-a$

1 *cBijB*

Note that $B^{-1} = B$. Pre multiply both sides by in equation (4) we get

0 *B*

$$\begin{bmatrix} 1 & -c_{ij}^{B} + c_{ij}^{B}B & -c_{ij}^{NB} + c_{ij}^{B}BN \\ 0 & B & N \end{bmatrix} \begin{bmatrix} z \\ x_{ij}^{B} \\ x_{ij}^{NB} \end{bmatrix} = \begin{bmatrix} c_{ij}^{B}B \begin{bmatrix} b \\ -a \end{bmatrix} \\ B \begin{bmatrix} b \\ -a \end{bmatrix} \end{bmatrix} \tag{5}$$

2.1. Analysis of Step by Step Procedure

For your convenience, we have included the algorithm stages in tabular form. Here is the simplex table formed according to equation set (5):

It is well understood that for the primal simplex approach to work, (zij − cij) < 0, and b > 0. Applying the dual simplex method reveals that none of the bs are equal to zero. We get the constraint equation from simplex table-1 as

$$Bx_{ij}^{B} + BNx_{ij}^{NB} = B \begin{bmatrix} b \\ -a \end{bmatrix}$$

Basic	*xBij*	*xNB*$_{ij}$	Solution
Z	*cBijB*−*cBij*	*cBijBN* −*cNBij*	$c_{ij}^{B}B \begin{bmatrix} b \\ -a \end{bmatrix}$
xBij	B	BN	$B \begin{bmatrix} b \\ -a \end{bmatrix}$

Table 1. Simplex

P_{ij}^{NB} and P_{ij}^{B} are the column vectors corresponding to matrix N and B. The constraint equation is associated with the l^{th} basic variable can be written as

$$\left(Bx_{ij}^{B}\right)_{l} + \sum_{i,j \in NB} \left(BP_{ij}^{NB}\right)_{l} x_{ij}^{NB} = \left(B \begin{bmatrix} b \\ -a \end{bmatrix}\right)_{l} \tag{6}$$

Because we are solving transportation problem using dual simplex algorithm first we select leaving variable. Suppose r^{th} basic variable is the leaving variable therefore corresponding constraint equation is written as:

$$\left(Bx_{ij}^{B}\right)_{r} + \sum_{i,j \in NB} \left(BP_{ij}^{NB}\right)_{r} x_{ij}^{NB} = \left(B \begin{bmatrix} b \\ -a \end{bmatrix}\right)_{r} \tag{7}$$

If P_{st}^{NB} is a vector corresponding to the entering variable x_{st}^{NB} then the r^{th} constraint equation can be written as follows.

$$\left(Bx_{ij}^{B}\right)_r + \sum_{i,j\in NB-st}\left(BP_{ij}^{NB}\right)_r x_{ij}^{NB} + \left(BP_{st}^{NB}\right)_r x_{st}^{NB} = \left(B\begin{bmatrix} b \\ -a \end{bmatrix}\right)_r$$

Thus entering variable $x^{NB}_{st}$is completed as

$$x_{st}^{NB} = \frac{\left(B\begin{bmatrix} b \\ -a \end{bmatrix}\right)_r}{(BP_{st}^{NB})_r} - \frac{(Bx_{ij}^{B})_r}{(BP_{st}^{NB})_r} - \frac{\sum_{i,j\in NB-st}(BP_{ij}^{NB})_r x_{ij}^{NB}}{(BP_{st}^{NB})_r} \tag{8}$$

Substituting the value ofx_{st}^{NB}in equation (6) we have

$$\left(Bx_{ij}^{B}\right)_l + \sum_{i,j\in NB-st}\left[\left(BP_{ij}^{NB}\right)_l - \left(BP_{st}^{NB}\right)_l \frac{(BP_{ij}^{NB})_r}{(BP_{st}^{NB})_r}\right] x_{ij}^{NB} - \left(BP_{st}^{NB}\right)_l \frac{(BP_{ij}^{B})_r}{(BP_{st}^{NB})_r}$$

$$= \left(B\begin{bmatrix} b \\ -a \end{bmatrix}\right)_l - \left(BP_{st}^{NB}\right)_l \frac{\left(B\begin{bmatrix} b \\ -a \end{bmatrix}\right)_r}{(BP_{st}^{NB})_r} \tag{9}$$

From the general matrix table, the z-equation coefficient associated with the variable x^{NB}_{ij} is $z_{ij} - c_{ij} = c_{ij}BN - c^{NB}_{ij}$.

Nothing that $(z_{ij} - c_{ij})$ is always zero for all basic variables x^{B}_{ij}.

$$z + \sum_{i,j\in NB}(z_{ij} - c_{ij})\, x_{ij}^{NB} = c_{ij}^{B} B\begin{bmatrix} b \\ -a \end{bmatrix} \tag{10}$$

i.e.,
$$z + \sum_{i,j\in NB-st}\left(c_{ij}^{B}BP_{ij}^{NB} - c_{ij}^{NB}\right) x_{st}^{NB} = c_{ij}^{B} B\begin{bmatrix} b \\ -a \end{bmatrix}$$

Substituting the value of x^{NB}_{st}(entering variable) from equation (8)

$$z + \sum_{i,j\in NB-st}\left[\left(c_{ij}^{B}BP_{ij}^{NB} - c_{ij}^{NB}\right) - \left(c_{ij}^{B}BP_{st}^{NB} - c_{st}^{NB}\right)\frac{(BP_{ij}^{NB})_r}{(BP_{st}^{NB})_r}\right] x_{ij}^{NB} - \left(c_{ij}^{B}BP_{st}^{NB} - c_{st}^{NB}\right)\frac{(Bx_{ij}^{B})_r}{(BP_{st}^{NB})_r}$$

$$= c_{ij}^{B} B\begin{bmatrix} b \\ -a \end{bmatrix} - \left(c_{ij}^{B}BP_{st}^{NB} - c_{st}^{NB}\right)\frac{\left(B\begin{bmatrix} b \\ -a \end{bmatrix}\right)_r}{(BP_{st}^{NB})_r} \tag{11}$$

Now we claim that$\left(BP_{st}^{NB}\right)_r$ is equal to -1. Substituting the value in equation (8), (9), (11) the equations are reduces.

With the help of the equations (8), (9), (11) transformed simplex table is given below

Basic	*xBij*	*xNBij*	Solution
Z	$\left(c_{ij}^{B}P_{st}^{NB} - c_{st}^{NB}\right)(B)_r$	P $_{i,j\in NB-(s,t)}\left[\left(c_{ij}^{B}BP_{ij}^{NB} - c_{ij}^{NB}\right) + \left(c_{ij}^{B}BP_{st}^{NB} - c_{st}^{NB}\right)\left(BP_{ij}^{NB}\right)_r\right]$	$c_{ij}^{B}B\begin{bmatrix} b \\ -a \end{bmatrix} + \left(c_{ij}^{B}BP_{st}^{NB} - c_{st}^{NB}\right)(B)_r\begin{bmatrix} b \\ -a \end{bmatrix}$

$(xBij)l$	$(B)_l + \left(BP_{st}^{N}\right)_l (B)_r$	$\sum_{(i,j)^{\in N-(s,t)}} \begin{bmatrix} (BP_{st}^{NB})_l + (BP_{st}^{NB})_l \\ \left(BP_{ij}^{NB}\right)_r \end{bmatrix}$	$\left(\begin{bmatrix} b \\ -a \end{bmatrix}\right)_l + (B)_r \begin{bmatrix} b \\ -a \end{bmatrix}$
$xNst$	$-(B)_r$	$\sum_{\sqrt{B-(s,t)}} (BP_{st}^{NB})_r$ $-$ $(i,j)\in$	$-(B)_r \begin{bmatrix} b \\ -a \end{bmatrix}$

Table 2. Simplex

With the help of simplex Table 1 and 2 we can solve the transportation problems.

2.2. Algorithm

Step 1: Rewrite the problem as L.P form.

Step 2: Write the standard form of the problem by adding surplus variables.

Step 3: Generate the table and find basic feasible solution.

Step 4: Generate the initial dual simplex table.

Step 5: Find leaving variable from table in step 4.

Step 6: Find entering variable from table in step 4.

Step 7: Generate the optimal dual simplex table described in the procedure.

Step 8: Write the optimal solution from table generated in step7.

3. Illustration

We consider a numerical example and give complete illustration of our algorithm. Consider the following transportation problem.

Destination				
Origin		D_1	D_2	Supply
	O_1	3	6	400
	O_2	4	5	300
	O_3	7	3	400
	Demand	450	350	

Step 1 : Rewriting the problem in the following form Maximize $Z^0 = -(3x_{11} + 6x_{12} + 4x_{21} + 5x_{22} + 7x_{31} + 3x_{32})$ subject to

$$x11 + x21 + x31 \geq 450$$

$$x12 + x22 + x32 \geq 350$$

$$-x_{11} - x_{12} \geq -400$$

$$-x_{21} - x_{22} \geq -300$$

$$-x_{31} - x_{32} \geq -400$$

Step 2 : Adding the surplus variables in each constraint

$$x11 + x21 + x31 - x01 = 450$$

$$x12 + x22 + x32 - x02 = 350$$

$$-x11 - x12 - x10 = -400$$

$$-x21 - x22 - x20 = -300$$

$$-x31 - x32 - x30 = -400$$

These equations can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x11 \\ x12 \\ x21 \\ x22 \\ x31 \\ x32 \\ x10 \\ x20 \\ x30 \\ x01 \\ x021 \end{bmatrix} = \begin{bmatrix} 450 \\ 350 \\ -400 \\ -300 \\ -400 \end{bmatrix}$$

Step 3 : Selecting the basic variable from table given below

	D_1	D_2	supply
O_1	3	6	400

	$x11$	$x12$	
O_2	4 $x21$	5 $x22$	300
O_3	7 $x31$	3 $x32$	400
Demand	450	350	

$c_{rj} = \min_{i=1,2,3}\{c_{ij}\},\ j = 1,2.\ c_{r1} = \min\{c_{il}\} = \min\{c_{11}, c_{21}, c_{31}\} = \min\{3,4,7\}$ = 3 for c_{11}, So x_{11} is the first basic variable, similarly x_{32} is another basic variable and other are surplus variables corresponding to supply constraints x_{10}, x_{20}, x_{30}.

Hence basic variables are x_{11}, x_{32}, x_{10}, x_{20}, x_{30}.

Step 4 : Now we construct the initial simplex table with the help of simplex table -1 as follows:

Basic	$x11$	$x32$	$x10$	$x20$	$x30$	$x12$	$x21$	$x22$	$x31$	$x01$	$x02$	Solution
Z	0	0	0	0	0	-3	-1	-2	-4	-3	-3	2400
$x11$	1	0	0	0	0	0	1	0	1	-1	0	450
$x32$	0	1	0	0	0	1	0	1	0	0	-1	350
$x10$	-1	0	-1	0	0	1	-1	0	-1	1	0	-50
$x20$	0	0	0	-1	0	0	1	1	0	0	0	300
$x30$	0	-1	0	0	-1	-1	0	-1	1	0	1	50

Step 5 : To find leaving vector, $(x^B_{ij})_r = \min[(x^B_{ij})_l, (x_{ij^B})_l < 0] = -50$. Here $r = 3$ indicates that the third basic variable x_{10} is the leaving variable so we remove x_{10} from the basis matrix.

Step 6 : To find the entering variable, $\min\left\{\frac{(c^B_{ij}BN - c^{NB}_{ij})}{(BN)_r}, (BN)_r < 0\right\} = \min\left\{\frac{-1}{-1}, \frac{-4}{-1}\right\}$ = 1. So x_{21} is the entering variable.

Step 7 : Using the procedure we get final(optimal) transformed Simplex Table 2 as follows:

Basic	$x11$	$x32$	$x10$	$x20$	$x30$	$x12$	$x21$	$x22$	$x31$	$x01$	$x02$	Solution
Z	1	0	1	0	0	-4	0	-2	-3	-4	-3	2400
$x11$	0	0	-1	0	0	1	0	0	0	0	0	450
$x32$	0	1	0	0	0	1	0	1	0	0	-1	350
$x21$	1	0	1	0	0	-1	1	0	1	-1	0	-50
$x20$	-1	0	-1	-1	0	1	0	1	-1	1	0	300
$x30$	0	-1	0	0	-1	-1	0	-1	1	0	1	50

Step 8 : All b are greater than zero so the optimal solution is obtained with the objective Z= 2450 with $x_{11} = 400$, $x_{32} = 350$, $x_{21} = 50$, $x_{20} = 250$, $x_{30} = 50$.

4. Conclusion

Complete details are provided on the dual matrix approach, which enables the algorithm to be modified for many types of transportation problems. It can help us figure out how to solve the transportation problem when there's more demand than supply. When developing algorithms to solve transportation problems with many goals, it could be useful.

References

[1] A. Charnes and W.W. Cooper's work in Management Science from 1954, titled "The Stepping-Stone approach for elucidating linear programming computations in transportation issues," is cited as [1]. [2] Ping JI and K.F. CHU's A Dual-Matrix Approach to the Transportation Problem was published in the Asia Pacific Journal of Operational Research in 2002 and can be found on pages 35–45. The authors of this work are S. Dharma and Kedar Nath Ram Nath from India's Operations Research group (2003).
The book "An Introduction to Operations Research" by Hamdy A. Taha was published in 2000 by Prentice-Hall in India.
A complex-type algorithm for general transportation problems as an alternative to stepping-stone, by H. Arsham and A.B. Kahn, Journal of the Operational Research Society, 40(1989), 581-590.
M.L. Balinski and R.E. Gomory's A Basic Approach to Assignment and Transportation Issues, published in Management Science in 1964, pages 578-593, is cited as [6]. [7] In their 1956 article "Transit Problem Solving," Ford and Fulkerson discuss the topic in Management Science, volume 3, pages 24-32. Journal of Operational Research Society, 35(1984), 1113-1114, by S.K. Goyal, enhances VAM for unbalanced transportation difficulties. Journal of Operational Research Society, 39(1988), 411-413, by A. Shafaat and S.K. Goyal, "Solution of Degeneracy in Transportation Issues" [3] [9].
[4]