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Using the principles of hyperbolic trigonometric functions, exact traveling wave solutions for the (2+1)-dimensional Konopelchenko-Dubrovsky equation

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Abstract

This study is the first to use the extended rational sinh-cosh method and the modified extended tanh-function method to obtain specific, appropriate solutions to the (2+1)-dimensional integro-differential Konopelchenko-Dubrovsky evolution equation through mathematical construction of traveling wave solutions. The above model was transformed into a nonlinear ordinary differential equation of third order using a traveling wave transformation. The model being studied has several complex hyperbolic trigonometric and rational functions that may be used to find soli-tary and periodic wave solutions. In order to emphasize the physical consequences of some of the solutions stated earlier, appropriate arbitrary parameters have been used to create two- and three-dimensional visualizations. The effect of evolution over time on the solution's structures has been shown in two-dimensional graphs.

Keywords: Inverse integral transformation, extended rational sinh-cosh method, modified extended tanh method, traveling wave transformation

1 Introduction

Remembering that accurate solutions to non-linear partial differential equations are highly important for highlighting many complicated and uncommon aspects in many areas of applied science is critical. In this particular case, symbolic computational software like Mathematica, Maple, and MATLAB greatly simplifies and simplifies the process of actively seeking new solutions for non-linear PDEs through the development of numerical and analytic methods. Several methods for solving non-linear partial differential equations have been refined throughout the years. The following evolution methods are used: the modified exponential function method [1], the Bernoulli sub-equation function methods [2, 3], the Laplace transformation method with the Caputo-fractional derivatives [4], the Hirota direct method [5, 6], the extended rational sin-cos and sinh-cosh methods [7, 8], the modified extended tanh-function method [9], the simplest equation method [10, 11], the techniques of the modified simplest equation and Sardar sub-equation techniques [12, 13], the Adomian decomposition method [14], the (1/G′)-expansion method, a modified version of the (1/G′) method, and many more. The integro-differential (2+1)-dimensional Konopelchenko-Dubrovsky equation that we examine possesses the this format:

The (2+1)-dimensional integro-differential Konopelchenko-Dubrovsky equation that we study has the following form:

$$u_t = u_{xxx} + 6\beta u u_x - \frac{3}{2}\alpha^2 u^2 u_x + 3\partial_x^{-1} u_{yy} - 3\alpha u_x \partial_x^{-1} u_y, \tag{1}$$

where α and β are arbitrary parameters, $u(x, y, t)$ is a sufficiently differentiable function and subscripts denoted the partial derivatives. The described configuration model in (1) is a distributed long-wave equation has been

created by Konopelchenko and Dubrovsky [15] and it was studied by using inverse scattering transform method. After using inverse integral transformation $\partial_x^{-1} = \int dx$ in (1), then one gets the equivalent system to (1) as follows:

$$\begin{aligned} u_t - u_{xxx} - 6\beta u u_x + \frac{3}{2}\alpha^2 u^2 u_x - 3Q_y + 3\alpha u_x Q = 0, \\ Q_x = u_y. \end{aligned} \tag{2}$$

In (1) if $u_t = 0$, then it turns into a well-known Gardner equation which is a combination form between KdV and modified KdV equations. Whereas if $\alpha = 0$, then (1) is converted into Kadomtsev-Petviashvili (KP) equation and if $\beta = 0$, it transformed into a modified version of the Kadomtsev-Petviashvili equation:

$$u_t = u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x + 3\partial_x^{-1} u_{yy} - 3\alpha u_x \partial_x^{-1} u_y.$$

Various researchers have studied different versions of the Konopelchenko-Dubrovsky equation (1) using different methods. These methods include the Sato theory and Hirota method, the Ja-cobi elliptic functions, the modified auxiliary equation technique, the truncated Painlevé expansion and the Möbious invariant form, the Hirota bilinear method with positive quartic-quadratic-functions, the improved of the extended F-expansion method, the modified simplest equation and cubic B-spline schemes, the Hirota direct method and linear superposition principle, and the references cited therein. A large number of scholars from different parts of the globe have looked at integralo-differential equations, specifically those that are partial or fractional differential equations, utilizing a variety of approaches, including [24–33]. The long tradition of analytical and semi-analytic approaches to solving mathematical issues is the main source of inspiration for the presented work. Because of the widespread use of partial and fractional differential equations in fields as diverse as biology, engineering, and physics, researchers from all over the world are eager to put these approaches to use. One more is that the inclusion of algorithms to back up the model gives readers faith that these tactics will provide satisfactory and accurate results.
The algorithmic flow of this work is as follows. Section 2 provides a concise overview of the approaches that were described. In section 3, proposed techniques are performed into the provided model and its generated solutions are illustrated visually. Lastly, final remarks were presented in the final part.

2 Descriptions of the methods

Usually, the construction and the structure of the mentioned methods commonly depend on the following step.

Step 1. Consider the following nonlinear PDE:

$$P(\psi, \psi_x, \psi_t, \psi_y, \psi_{xt}, \psi_{xx}, \psi_{yt}, \psi_{xyt}, \cdots) = 0, \tag{3}$$

where $\psi = \psi(x, y, t)$. Setting

$$\psi(x, y, t) = U(Q), \; Q = \delta_1 x + \delta_2 y + \delta_3 t, \tag{4}$$

where δ_i for $i = 1, 2, 3$ are arbitrary parameters distinct from zero. They are acting as indicators of the horizontal velocity, amplitude of the motion and viscosity of the wave propagation speed correspondingly. If (4) is substituted in (3), then one obtains

$$N(U, U', U'', \cdots) = 0, \tag{5}$$

where $U = U(Q)$, $U' = \frac{dU}{dQ}$, $U'' = \frac{d^2U}{dQ^2}, \cdots$.

The following two selected techniques are successful and trustworthy for assessing the perfect approach of the well-known mathematical models.

2.1 Structure of the extended rational sinh-cosh method (ERSCM)

This method is described in the following steps where step one is identically the same as the first step in the above.

Step 2. Let (5) has solutions with the following forms:

$$\psi(Q) = \frac{\gamma_0 \sinh(\mu Q)}{\gamma_2 + \gamma_1 \cosh(\mu Q)}, \cosh(\mu Q) \neq -\frac{\gamma_2}{\gamma_1}, \tag{6}$$

or in the form,

$$\psi(Q) = \frac{\gamma_0 \cosh(\mu Q)}{\gamma_2 + \gamma_1 \sinh(\mu Q)}, \sinh(\mu Q) \neq -\frac{\gamma_2}{\gamma_1}, \tag{7}$$

where γ_0, γ_1 and γ_2 are purposeful coefficients to be determined later and μ is a non-zero wave number.

Step 3. Anonymous parameters can be discovered by replacing one of (6) or (7) into (5), assembling all the terms which have the same powers of $\cosh^{\lambda}(\mu Q)$ or $\sinh^{\lambda}(\mu Q)$ and equating to zero all the coefficients of $\cosh^{\lambda}(\mu Q)$ or $\sinh^{\lambda}(\mu Q)$ produces several algebraic equations. It is conceivable to find out the solutions to the system of algebraic equations utilizing various packages of symbolic computing tools.

Remark 1. Alternatively, one may gather and equalize to zero all the parameters for the same powers of $\cosh^{\lambda 1}(\mu Q)\sinh^{\lambda 2}(\mu Q)$, it generates a system of the algebraic equations, therefore solving the obtained system, where $\lambda_1, \lambda_2 = 0, 1, 2, \cdots, n$.

Step 4. By restoring the values of γ_0, γ_1, γ_2 and μ into (6) or (7), the solution of (5) can be found, combining parameters, and re-installing obtained solutions one can derive the solution for the (3).

2.2 Structure of the modified extended tanh method (METM)

The METM is abbreviated in the following steps where step one is the same as step one in the above.

Step 2. Assume that (5) has the following as the possible solutions

$$U(Q) = a_0 + \sum_{i=1}^{m} (a_i \varphi^i + b_i \varphi^{-i}), \tag{8}$$

where a_0, a_i, b_i, $i = 1, \cdots, m$ are constants has to be demonstrated later such that either $a_m \neq 0$ or $b_m = 0$ and the function $\varphi(Q)$ satisfies the given Riccati ordinary differential equation

$$\varphi'(Q) = b + \varphi^2(Q), \tag{9}$$

solution of (9) are publicized as.

Case 1: If $b < 0$, then

$$\varphi(Q) = -\sqrt{-b}\tanh(\sqrt{-b}Q), \text{ or } \varphi(Q) = -\sqrt{-b}\coth(\sqrt{-b}Q). \quad (10)$$

Case 2: If $b > 0$, then

$$\varphi(Q) = \sqrt{b}\tan(\sqrt{b}Q), \quad \text{or} \quad \varphi(Q) = -\sqrt{b}\cot(\sqrt{b}Q). \tag{11}$$

Case 3: If $b = 0$, then

$$\varphi(Q) = -\frac{1}{Q}. \tag{12}$$

Step 3. The positive integer m should be determined in (5) by applying the balancing principle between the greater power of the nonlinear terms and the highest order derivatives.

Step 4. Putting (8) and (9) into (5) and combining all the phrases with the same power of φ^{i} equating to zero for $i = 0, \mp1, \mp2, \cdots$Finally a family of algebraic equations solved by using computer packages to obtain values of these parameters.

Step 5. Contracted values of constants, and using equations (10)-(12) one can obtain the exact solution of (3).

3 Applications of the proposed methods

Applying wave transformation (4) to (2), after some mathematical simplifications then one obtains:

$$\frac{\delta_3\delta_1 - 3\delta_2}{\delta_1}\left(U' + 3\alpha\delta_2^2 - 6\beta\delta_1\right)UU' + \frac{3}{2}\delta_1\alpha^2U^2U' - \delta_1^3U''' = 0, \tag{13}$$

taking integration once where assume $U(\cdot)$ at initial point is zero, then one ends up with

$$2(\delta_3\delta_1 - 3\delta_2)U' + \delta_1\left(3\alpha\delta_2^2 - 6\beta\delta_1\right)U^2 + \delta_1^2\alpha^2U^3U' - 2\delta_1^4U'' = 0. \tag{14}$$

The ordinary differential equation (14) is an approved equation to performing the balance principle.

3.1 Application of ERSCM to the aforementioned model

To solve (2) by using ERSCM first assume that (14) have the solutions of the following form:

$$U(Q) = \frac{\gamma_0\sinh(\mu Q)}{\gamma_2 + \gamma_1\cosh(\mu Q)}, \cosh(\mu Q) \neq -\frac{\gamma_2}{\gamma_1}, \tag{15}$$

where γ_i for $i = 0, 1, 2$ are parameters with $\gamma_0 \neq 0$, $\gamma_1^2 + \gamma_2^2 \neq 0$ and also μ is a non-zero wave number. Subsequently, the first and second derivatives of the assumed trial solution in (15) with respect to Q are taken the following format:

$$U'(Q) = \frac{\gamma_0\mu\left(\gamma_2\cosh(\mu Q) + \gamma_1\right)}{\left(\gamma_1\cosh(\mu Q) + \gamma_2\right)^2}, \tag{16}$$

and

$$U''(Q) = \frac{\gamma_0\mu^2\sinh(\mu Q)\left(-\gamma_2\gamma_1\cosh(\mu Q) - 2\gamma_1^2 + \gamma_2^2\right)}{\left(\gamma_1\cosh(\mu Q) + \gamma_2\right)^3}. \tag{17}$$

Subbing (15)- (17) into the transformed ODE (14), one gets the following

$$\begin{aligned}&\alpha^2\gamma_0^2\delta_1^2\cosh^2(\mu Q) + 3\alpha\gamma_0\gamma_2\delta_2\delta_1\cosh(\mu Q) + 3\alpha\gamma_0\gamma_1\delta_2\delta_1\sinh(\mu Q)\cosh(\mu Q)\\ &- 6\beta\gamma_0\gamma_2\delta_1^2\cosh(\mu Q) - 6\beta\gamma_0\gamma_1\delta_1^2\sinh(\mu Q)\cosh(\mu Q) + 2\gamma_1\gamma_2\delta_1^4\mu^2\sinh(\mu Q)\\ &- 4\gamma_1\delta_1^4\mu^2 - 2\gamma_2\delta_1^4\mu^2 + 2\gamma_1\delta_3\delta_1\sinh^2(\mu Q) - 6\gamma_1^2\delta_2\sinh^2(\mu Q) + 2\gamma_2\delta_3\delta_1\end{aligned} \tag{18}$$

$$-12\gamma_1\gamma_2\delta_2 \sinh(\mu Q) + 4\gamma_1\gamma_2\delta_3\delta_1 \sinh(\mu Q) - 6\gamma_2 \delta_2 = 0.$$

In (18) collecting all the terms with the same powers of $\cosh^{\tau_1}(\mu Q)\sinh^{\tau_2}(\mu Q)$ where $\tau_1, \tau_2 = 0, 1, 2$ and equating to zero. The coefficients of $\cosh^{\tau_1}(\mu Q)\sinh^{\tau_2}(\mu Q)$ yields a set of algebraic equations as in the following:

$$\begin{aligned}
&\alpha^2\gamma_0^2\delta_1^2 - 4\gamma_1^2\delta_1^4\mu^2 - 2\gamma_2^2\delta_1^4\mu^2 - \delta_3\delta_1 + 3\delta_2^2 = 0,\\
&2\gamma_1\gamma_2\delta_1^4\mu^2 + 2\delta_3\delta_1 - 6\delta_2^2 = 0,\\
&\alpha^2\gamma_0^2\delta_1^2 + 2\gamma_1^2\delta_1\delta_3 - 3\delta_2^2 = 0,\\
&3\gamma_0\gamma_2\delta_1(\alpha\delta_2 - 2\beta\delta_1) = 0,\\
&3\gamma_0\gamma_1\delta_1(\alpha\delta_2 - 2\beta\delta_1) = 0.
\end{aligned} \tag{19}$$

Solving (19) by the computer's software packages, outcomes are the following parameters.

Case 1. Obtained parameters in solving algebraic system (19) are in the following

$$\gamma_2 = i\gamma_1,\ \alpha = \frac{\gamma_1\delta_1\mu}{\gamma_0},\delta_2 = \frac{2\beta\gamma_0}{\gamma_1\mu},\delta_3 = \frac{24\beta^2\gamma_0^2 - \gamma^2\delta_1^4\mu^4}{2\gamma_1^2\delta_1\mu^2}. \tag{20}$$

Substituting parameters (20) with (15) into (14), the solution of (2) has been derived as follows

$$u_{S1} = \frac{\gamma_0\cosh\left(\frac{12\beta^2\gamma_0^2 t}{\gamma_1^2\delta_1\mu} - \frac{1}{2}\delta_1^3\mu^3 t + \delta_1\mu x + \frac{2\beta\gamma_0 y}{\gamma_1}\right)}{\gamma_1\sinh\left(\frac{12\beta^2\gamma_0^2 t}{\gamma_1^2\delta_1\mu} - \frac{1}{2}\delta_1^3\mu^3 t + \delta_1\mu x + \frac{2\beta\gamma_0 y}{\gamma_1}\right) + i}. \tag{21}$$

Graphs of the solution in (21) where $y = \frac{2}{3}, \mu = \frac{6}{5}, \beta = \frac{1}{3}, \gamma_0 = -\frac{3}{2}, \gamma_1 = \frac{9}{4}, \delta_1 = \frac{3}{2}$ and $-10 \le x \le 10, -10 \le t \le 10$ are given in the following:

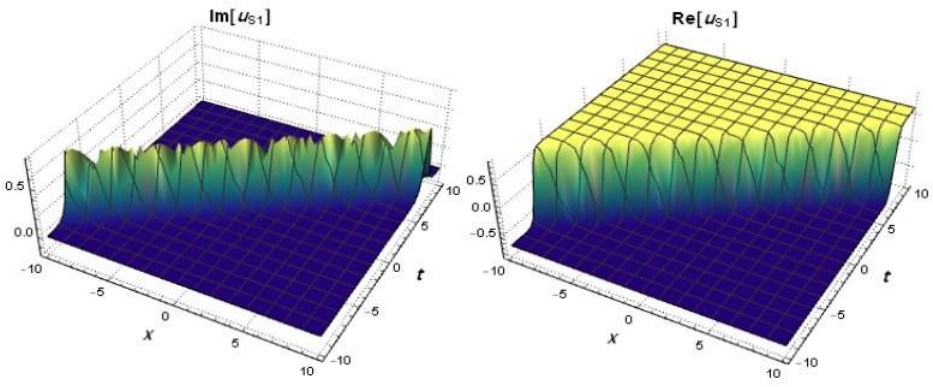


Fig. 1 3D figures of (21).

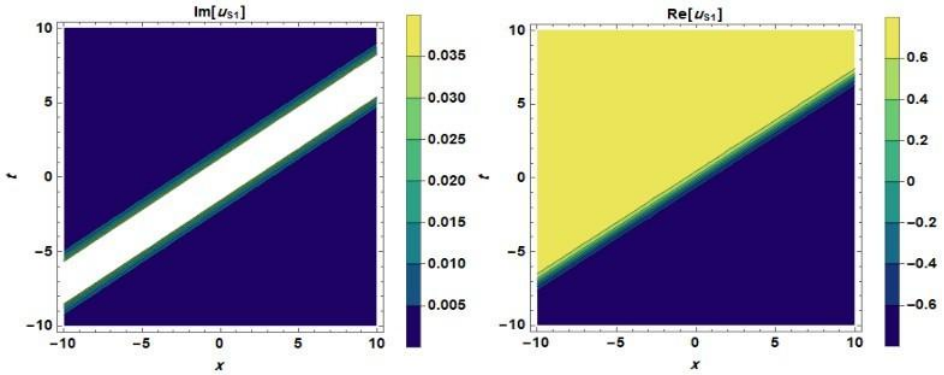


Fig. 2 Contour surfaces of (21)

The influence of time evolution on the solution's features is presented by the following two-dimensional graphs where in the legend t values are mentioned.

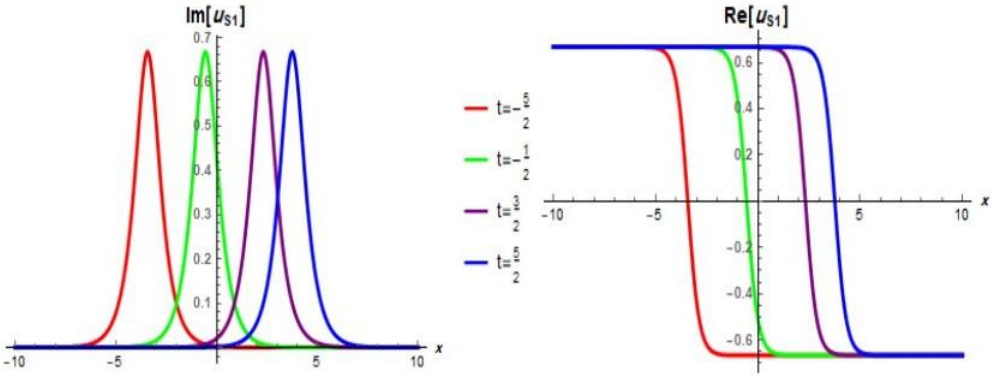


Fig. 3 2D graphs of (21).

Case 2. By solving (19) one obtains

$$\gamma_1 = i\gamma_2, \alpha = \frac{i\gamma_2\delta_1\mu}{\gamma_0}, \beta = \frac{i\gamma_2\delta_2\mu}{2\gamma_0}, \delta_3 = \frac{6\delta_2^2 - \delta_1^4\mu^2}{2\delta_1}. \tag{22}$$

By re-installing obtained parameters in (22) with (15) into (14), the following solution of (2) are obtained

$$u_{S2} = \frac{\gamma_0 \cosh\left(-\frac{1}{2}\delta_1^3\mu^3 t + \frac{3\delta_2^2\mu t}{\delta_1} + \delta_1\mu x + \delta_2\mu y\right)}{\gamma_2\left(1 + i\sinh\left(-\frac{1}{2}\delta_1^3\mu^3 t + \frac{3\delta_2^2\mu t}{\delta_1} + \delta_1\mu x + \delta_2\mu y\right)\right)}. \tag{23}$$

Case 3. The following parameters are obtained by solving (19)

$$\gamma_2 = 0, \alpha = \frac{2\gamma_1\delta_1\mu}{\gamma_0}, \delta_2 = \frac{\beta\gamma_0}{\gamma_1\mu}, \delta_3 = \frac{3\beta^2\gamma_0^2 - 2\gamma_1^2\delta_1^4\mu^4}{\gamma_1\delta_1\mu^2}. \tag{24}$$

Installing obtained parameters in (24) with (15) into (14), the solution of (2) has been found as follows

$$u_{S3} = \frac{\gamma_0}{\gamma}\coth\left(\frac{3\beta^2\gamma_0^2 t}{2} - 2\delta_1^3\mu^3 t + \delta_1\mu x + \frac{\beta\gamma_0 y}{\gamma}\right). \tag{25}$$

1 $\gamma_1 \delta_1 \mu$ 1

Remark 2. Correspondingly by supposing that (14) have the solutions in the form of

$$u(Q) = \frac{\gamma_0 \cosh(\mu Q)}{\gamma_2 + \gamma_1 \sinh(\mu Q)}, \sinh(\mu Q) \neq -\frac{\gamma_2}{\gamma_1},$$

where γ_i for $i = 0, 1, 2$ are parameters to be determined where γ_0 different from zero, γ_1, γ_2 not both are zero and μ is a non-zero wave number. With the same previous arguments one gets some other solutions of (2).

3.2 Application of METM to the aforementioned model

For solving model in (2) with the METM firstly applying the balance principle in (14) between U'' and U^3 one directly gets $m = 1$ and (8) becomes

$$U = a_0 + a_1 R + \frac{b_1}{R}, \tag{26}$$

where a_0, a_1, b_1 are constants to be determining later, $a_1 \neq 0$ or $b_1 = 0$, here R is a function satisfies the following Riccati ODE

$$R' = b + R^2. \tag{27}$$

Respectively, taking first and second derivatives in (26) by considering (27), one directly gets

$$U' = a_1 b - b_1 + a_1 R^2 - \frac{b b_1}{R^2}, \tag{28}$$

and

$$U'' = 2a_1 bR + 2a_1 R^3 + \frac{2b^2 b_1}{R^3} + \frac{2bb_1}{R}. \tag{29}$$

Inserting (26)-(29) into (14) after simplification collect all the coefficients of R^i where $i = 0, 1, \ldots, 6$ and equating the coefficients of the same power of R^i to zero. A system of algebraic equations for the parameters as follows

$$\begin{aligned}
&\alpha^2 b_1^3 \delta_1^2 - 4b^2 b_1 \delta_1^4 = 0,\\
&\alpha^2 a_1^3 \delta_1^2 - 4a_1 \delta_1^4 = 0,\\
&3\alpha^2 a_0 b_1^2 \delta_1^2 + 3\alpha b_1^2 \delta_1 \delta_2 - 6\beta b_1^2 \delta_1^2 = 0,\\
&3\alpha^2 a_0 a_1^2 \delta_1^2 + 3\alpha a_1^2 \delta_1 \delta_2 - 6a_1^2 \beta \delta_1^2 = 0,\\
&3\alpha^2 a_1 b_1^2 \delta_1^2 + 3\alpha^2 a_0^2 b_1 \delta_1^2 + 6\alpha a_0 b_1 \delta_2 \delta_1 - 12 a_0 \beta b_1 \delta_1^2\\
&- 4bb_1 \delta_1^4 + 2b_1 \delta_3 \delta_1 - 6b_1 \delta_2^2 = 0,\\
&\alpha^2 a_1^3 \delta_1^2 + 3\alpha a_0^2 \delta_1 \delta_2 + 6\alpha^2 a_1 a_0 b_1 \delta_1^2 + 6\alpha a_1 b_1 \delta_1 \delta_2\\
&- 12 a_1 \beta b_1 \delta_1^2 - 6a_0^2 \beta \delta_1^2 - 6a_0 \delta_2^2 + 2a_0 \delta_1 \delta_3 = 0.\\
&3\alpha^2 a_0^2 a_1 \delta_1^2 + 6\alpha a_0 a_1 \delta_2 \delta_1 + 3\alpha^2 a_1^2 b_1 \delta_1^2 - 4a_1 b \delta_1^4\\
&- 12 a_0 a_1 \beta \delta_1^2 + 2a_1 \delta_3 \delta_1 - 6a_1 \delta_2^2 = 0.
\end{aligned} \tag{30}$$

The system (30) is solvable via computer programs and the following cases are obtained.

Case 1. Captured parameters in solving (30) are in the following:

$$\beta = -\frac{\sqrt[3]{\delta_1 \left(16 b \delta_1^3 + \delta_3\right) + 12i \sqrt{b}\, \delta_1^2}}{\sqrt[3]{\delta_1 \left(16 b \delta_1^3 + 3a_1\right)}}, \alpha = \frac{2\delta_1}{a_1},$$
$$\delta_2 = -\frac{\delta_1 \left(16 b \delta_1^3 + \delta_3\right)}{3a_0} = -2i a_1 \sqrt{b}, b_1 \tag{31}$$

$= -ba_1$.

The coefficients in (31) and for the different sign of b the following sub-cases of the solutions are received.

Case 1.1. Where $b < 0$ and performing $R = -\frac{\sqrt{-b}\tanh\sqrt{-bQ}}{}$ as a solution of the Riccati ODE (27) the obtained solution of (2) becomes

$$u_{11} = -a_1\left(\sqrt{b}\tan\left(\sqrt{b}(Ly+\delta_3 t+\delta_4 x)\right)\cot\left(\sqrt{b}(Ly+\delta_3 t+\delta_4 x)\right)+i\right)^2, \tag{32}$$

here $L = -\frac{\sqrt{\delta_1}\sqrt{16b\delta_1^3+\delta_3}}{\sqrt{3}}$.

Case 1.2. If $b > 0$, and $R = \sqrt{b}\tan\sqrt{bQ}$ denoted as a solution of the Riccati ODE (27), then obtained solution of (2) becomes

$$u_{12} = -a_1\left(\sqrt{b}\tan\left(\sqrt{b}L\right)\cot\left(\sqrt{b}L\right)+i\right)^2, \tag{33}$$

where $L = \delta_1 x - \frac{\sqrt{\delta_1}\sqrt{16b\delta_1^3+\delta_3}}{\sqrt{3}}y+\delta_3 t$. Profile of the solution in (33) has been appointed in the following where $a_1 = -\frac{3}{2}$, $\alpha = \frac{1}{5}$, $\delta_1 = -\frac{2}{3}$, $\delta_3 = \frac{2}{3}$, $b = \frac{1}{16}$, $y = \frac{3}{5}$, and $-10 \le x \le 10$, $-10 \le t \le 10$.

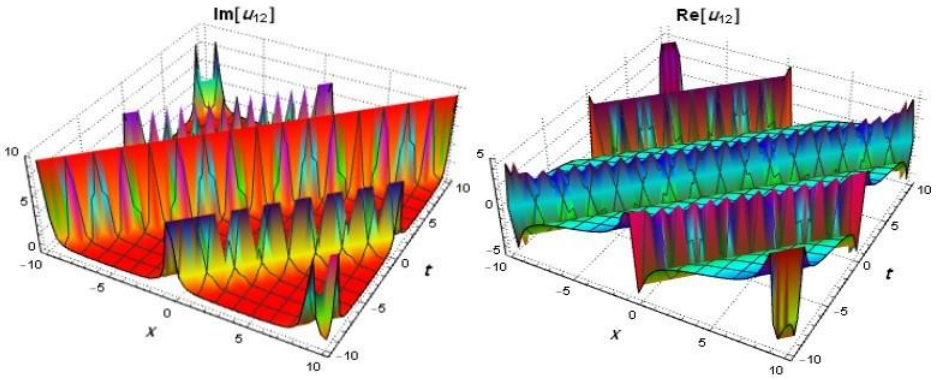


Fig. 4 3D figures of (33).

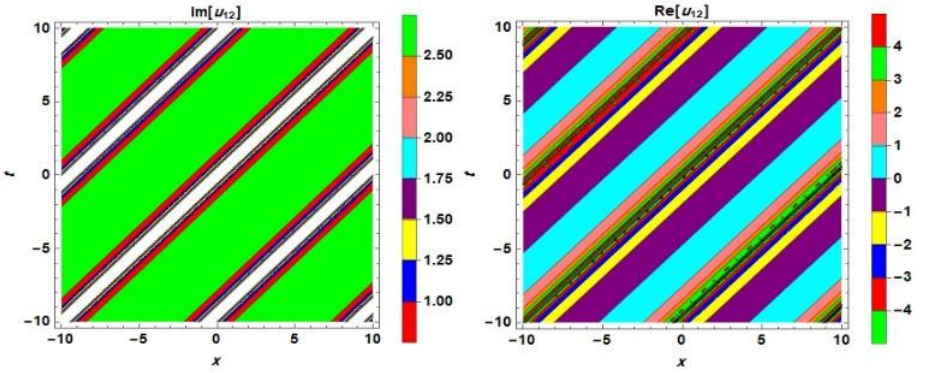


Fig. 5 Contour surfaces plot of (33).

Where the t values are given in the legend and $-50 \le x \le 50$. We have

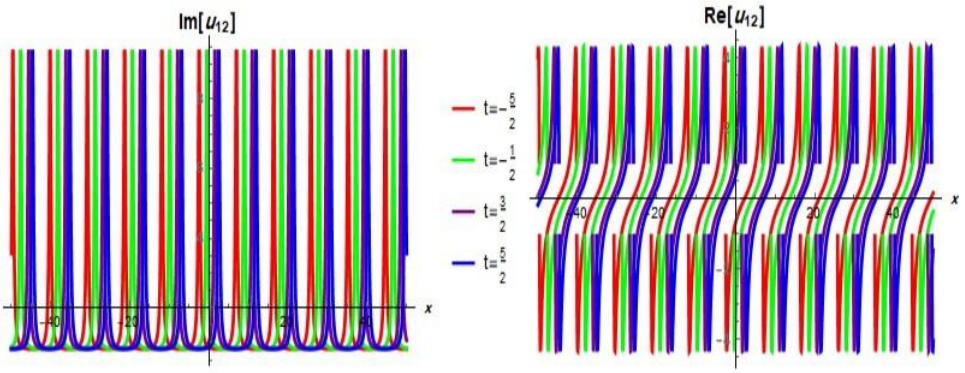


Fig. 6 2D graphs of (33).

Graphs that are displayed in Figure 6 demonstrate the influence of the time development on the framework of the obtained solution note that t values are indicated in the legend. In the following three-dimensional figures $t = \frac{3}{2}$, $-10 \le x \le 10$.

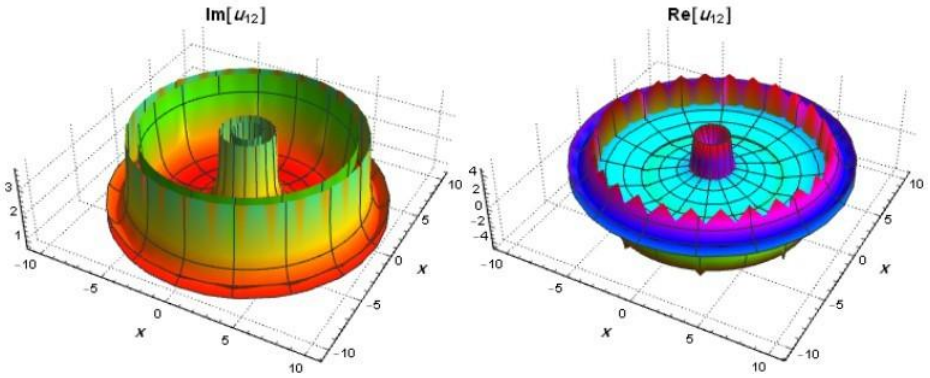


Fig. 7 3D revolving plot of (33).

Case 1.3. If $b = 0$ and using $R = -\frac{1}{Q}$ as a solution of the Riccati ODE (27), then obtained solution of the (2) take the following formation:

$$u_{13} = -\frac{3a_1}{3\delta_3 t + 3\delta_1 x - \sqrt{3}\sqrt{\delta_1}\sqrt{\delta_3}y}.$$

Remark 3. Here the significance of picking parameter values is obviously apparent. They determine features and physical aspects of the generated solutions.

Case 2. The following parameters are obtained:

$$\beta = -\frac{\delta_2}{a_1} + i\alpha^2 a_1 \sqrt{b}, \delta_3 = 2\alpha^3 a_1 b - \frac{6\delta_2^2}{\alpha a_1}, a_0 = 2ia_1 \sqrt{b}, b_1 = -a_1 b, \delta_1 = -\frac{1}{2}\alpha a_1. \quad (34)$$

Coefficients in (34) and sign of the constant b has a main role to generate the following sub-cases of the solutions.

Case 2.1. If $b < 0$ and using $R = -\sqrt{-b}\tanh\sqrt{-bQ}$ as a solution of the Riccati ODE (27). Then one gets

the following solution of (2)

$$u_{21} = a_1 \sqrt{\frac{b}{}} \tan\left(\sqrt{\frac{b}{}} L\right) - \cot\left(\sqrt{\frac{b}{}} L\right) + 2i, \tag{35}$$

where

$$L = -\frac{1}{2}\alpha a_1 x + \delta_2 y + \left(2\alpha^3 a_1^3 b - \frac{6\delta_2^2}{\alpha a_1}\right) t.$$

Case 2.2. If $b > 0$ using $R = \sqrt{\frac{b}{}} \tan\left(\sqrt{\frac{b}{}} Q\right)$ as a solution of the Riccati ODE (27), then the following solution of (2) has been captured as:

$$u_{22} = a_1 \sqrt{\frac{b}{}} \tan\left(\sqrt{\frac{b}{}} L\right) - \cot\left(\sqrt{\frac{b}{}} L\right) + 2i, \tag{36}$$

where

$$L = -\frac{1}{2}\alpha a_1 x + \delta_2 y + \left(2\alpha^3 a_1^3 b - \frac{6\delta_2^2}{\alpha a_1}\right) t.$$

The figures of the solution in (36) presented in the following for the mentioned values $a_1 = \frac{5}{2}, \alpha = \frac{5}{3}, \delta_1 = -\frac{1}{3}, \delta_2 = -\frac{2}{3}, b = \frac{1}{50}, y = -5$ and $-10 \le x \le 10, -10 \le t \le 10$.

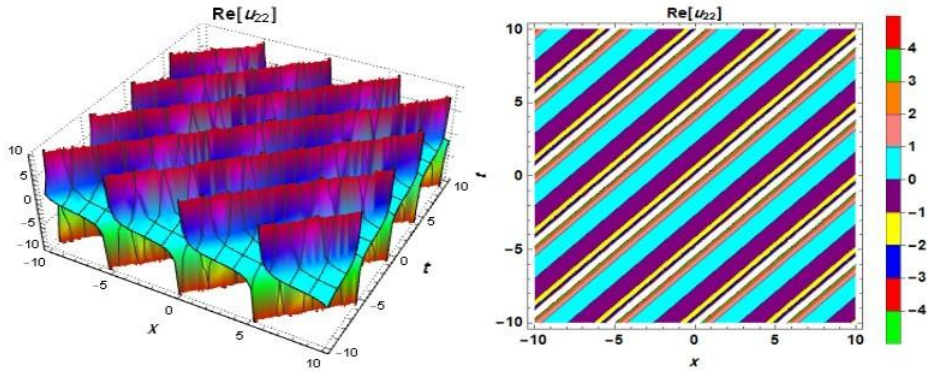


Fig. 8 3D figures and contour surface of (36).

In the following $-20 \le x \le 20$, for 2D graphs t are given in the legend. For 3D figure choosing $t = \frac{1}{2}$, $-10 \le x \le 10$.

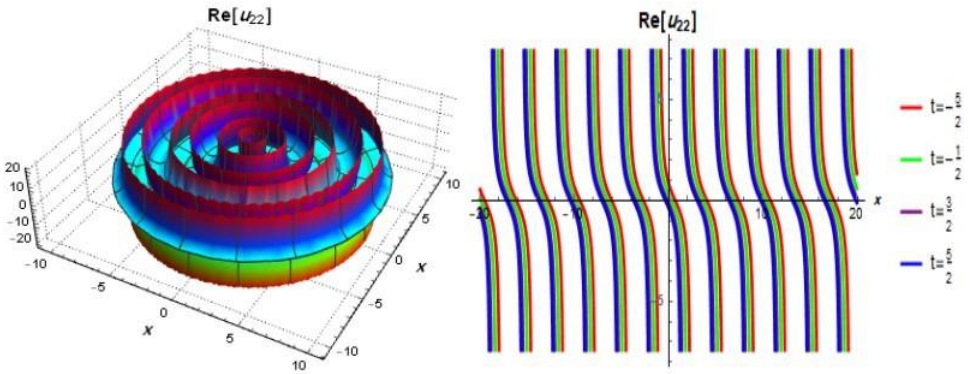


Fig. 9 2D graphs and 3D revolving plot of (36).

Case 2.3. If $b = 0$ and using $R = -\frac{1}{Q}$ as a solution of the Riccati ODE (27), then obtained solution of the (2) take the subsequent structure:

$$u_{23} = -\frac{2\alpha a_1^2}{\alpha^2 a_1^2 x + 2\alpha a_1 \delta_2 y + 12\delta_2^2 t}.$$

Case 3. The following are clutched coefficients:

$$a_0 = \frac{ib_1}{\sqrt{b}}, a_1 = 0, \alpha = -\frac{2b\delta_1}{b_1}, \beta = \frac{-b\delta_2 + 2ib^{3/2}\delta_1^2}{b_1}, \delta_3 = \frac{3\delta_2^2}{\delta_1} - 4b\delta_1^3. \tag{37}$$

Parameters in (37) are caused the following sub-cases of solutions of (2) by taking into consideration the sign of b.

Case 3.1. If $b < 0$ using $R = -\sqrt{-b}\tanh\sqrt{-bQ}$ as a solution of the Riccati ODE in (27), then one obtains the following solution of (2)

$$u_{31} = b_1\left(-\frac{\sqrt{-b}\coth\left[\sqrt{-b}\left(t\left(\frac{3\delta_2^2}{\delta_1} - 4b\delta_1^3\right) + \delta_1 x + \delta_2 y\right)\right]}{\sqrt{-b}} + \frac{i}{b}\right). \tag{38}$$

Case 3.2. If $b > 0$ using $R = \sqrt{b}\tan\sqrt{bQ}$ as a solution of the Riccati ODE (27), then the obtained solution of (2) becomes

$$u_{32} = \frac{b_1\left(\cot\left[\sqrt{b}\left(-4b\delta_1^3 t + \frac{3\delta_2^2 t}{\delta_1} + \delta_1 x + \delta_2 y\right)\right] + i\right)}{\sqrt{b}}, \tag{39}$$

where $b_1 = \frac{1}{2}, \delta_1 = -\frac{2}{3}, \delta_2 = \frac{2}{3}, b = \frac{3}{2}, y = \frac{3}{5}$ and $-10 \le x \le 10, -10 \le t \le 10$. The solution in (39) has the following figures:

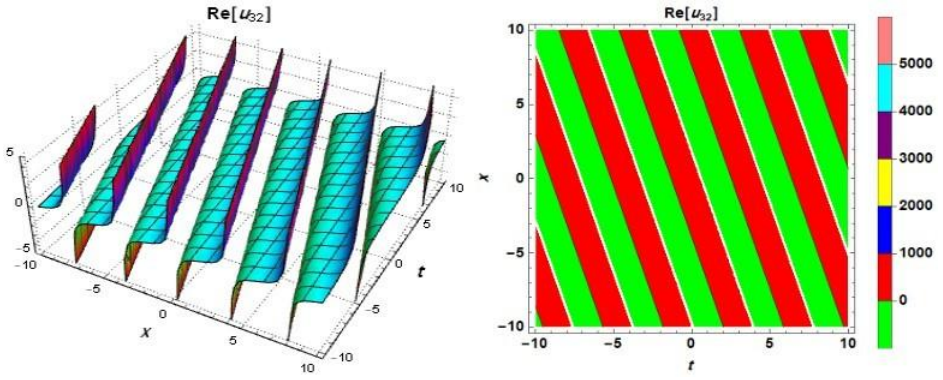


Fig. 10 3D figures and contour surface of (39).

In the following $-10 \leq x \leq 10$, for 2D graphs t are given in the legend and for 3D figure fixing $t = \frac{1}{2}$.

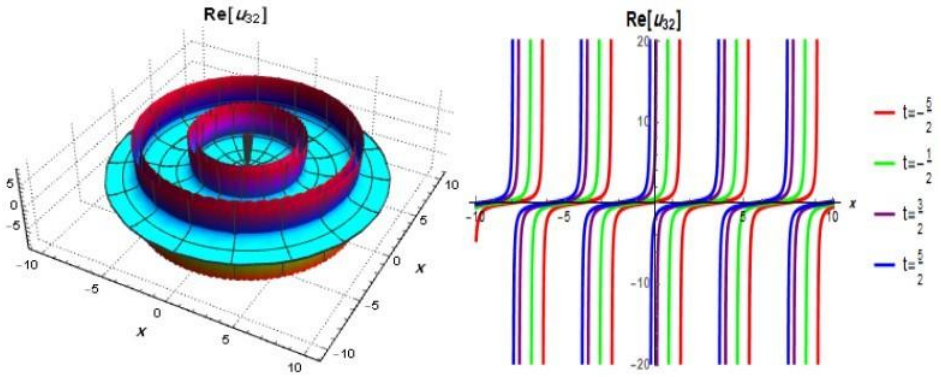


Fig. 11 2D graphs and 3D revolving plot of (39).

Remark 4. A zero value of *b* is impossible since it renders some of the coefficients undefined and so in this scenario, one may claim (2) has only a trivial solution.

4 Conclusion

This work investigates certain traveling wave solutions by using the ERSCM and METM to the (2+1)-dimensional non-linear integro-differential Konopelchenko-Dubrovsky evolution problem for the first time. Results are obtained by use of rational, osculating, hyperbolic trigonometric, and solitonic functions. The answers for the outcomes are also displayed visually. Additionally, Figures (3), (6), (9) and (11) show the consequences of the temporal evolution in two-dimensional graphs that may be seen visualy. The study's findings attest to the techniques' efficacy and dependability. In addition, the results show that the free parameters of the solution significantly affect the waveform and its behavior, suggesting that these factors may stand in for a wide range of complex, fresh features seen in many scientific disciplines. Several novel analytical solutions are introduced for the first time, and our results are novel and considerable in comparison to other prior studies. Furthermore, all of the produced solutions have been verified to fulfill the corresponding equation utilizing symbolic computational software tools.

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