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Finding the analytical solution to the fractional order Pseudo-Hyperbolic Telegraph equation by use of the modified double Laplace transform technique

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Abstract

Using the Caputo fractional derivative as its foundation, this work investigates the Pseudo-Hyperbolic Telegraph partial differential equation (PHTPDE). Particularly for this model, we build the modified double Laplace transform technique (MDLTM). The analytic solution for the fractional-order pseudo-hyperbolic telegraph problem described by the Caputo derivative was obtained using the MDLTM. For the fractional-order pseudo-hyperbolic telegraph equation, the suggested method offers an extremely efficient analytical approach. An example test issue was given. Looking at the findings, it's obvious that this approach is more convenient and uses less steps to get the same analytical solution as other methods. This work asserts that the fractional order pseudohyperbolic telegraph equation order can be analytically solved using the MDLTM. You may represent an analytical solution in terms of mathematical functions or known procedures since it leads to an exact, closed-form answer. This is an impressive feat because finding analytical solutions to partial differential equations (PDEs) is not always easy, particularly when dealing with fractional order problems.

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Keywords: Pseudo-hyperbolic telegraph equation, caputo fractional derivative, modified double Laplace transform method, analytical solution.

1 Introduction

Fractal calculus is the branch of mathematics concerned with integrals and differentials of non-integer order and their applications. When comparing accuracy and efficiency, fractional models are superior to classical ones. In addition, it has been found that noninteger derivative operators are critical for comprehending physical events [1–5]. Numerous disciplines have explored the benefits of fractional calculus, including mechanics, engineering, biology, control theory, medicine, and seismology [6-9]. A high-order partial differential equation with mixed partial derivative is the PHTPDE that is based on the Caputo derivative. Longitudinal vibrations, nerve conduction, plasma physics, and many other physical phenomena may be explained by the famous math-ematical physics equation that underpins the suggested paradigm. There have been a number of publications written about the pseudo-hyperbolic equation's behavior. Some of these works address the uniqueness and existence of solutions, analytical and approximate solutions, numerical calculations, stability estimations, and others. difficulty with the pseudo-hyperbolic equation of the Cauchy type, and whether or not a solution exists [13]. Recent years have seen a proliferation of efficient methods for approximating and solving fractional-order partial differential equations, such as the double Laplace transform method [14], theta-method [15], the finite element method [17], the Laplace de-composition method [18], the fractional sub equation method [19], MDLTM [20], and the Fibonacci wavelet method [21]. When dealing with multi-state neutral fractional order systems that are susceptible to impulsive perturbations and state delays, the authors of [22] investigated their finite temporal stability (FTS). A non-local Sobolev type

fractional differential equation with impulsive circumstances was discussed in [23], and for each approximation, the presence and uniqueness of mild solutions were investigated using the Banach fixed point approach and analytic semigroup. Next, in order to determine particular convergence results for approximative solutions, the Faedo-Galerkin approximation was used. In [24], the researchers investigated if there are any solutions and if they are unique to the T-Cauchy issue, which involves neutral functional sequential integro-differential equations with a Caputo fractional derivative on time scales. In order to solve several key models that arise in real-world issues, the researchers in [25,26] used the q-homotopy analysis transform approach.

The authors in [20] used the double conformable Laplace transform method to arrive at an analytical solution to a fractional-order partial differential equation; this is in contrast to the widespread use of the double Laplace transform method in academia, which is a powerful tool for solving fractional-order partial differential equations. The authors of [27] proposed the (DLDM) and the (FDM), which were used analytically and roughly to solve the time-dependent (FSPPPD) Equation with Caputo derivative. With its many benefits, the modified double Laplace transform technique is a game-changing approach. Compared to current methods, which require more procedures to accomplish the same goal, it is more practical and requires fewer processes to deliver an analytical answer. Using the Caputo fractional derivative as a foundation, this work examines the following PHTE [29] of fractional order.

$$u_{tt}(x,t) + D^{\alpha}_{t} u(x,t) + u(x,t) - \lambda u_{txx}(x,t) - u_{xx}(x,t) = f(x,t), \quad (1)$$

$$0 < x < L, t > 0,\ \lambda > 0,\ 0 < \alpha \le 1,$$

$$u(x,0) = h(x),\ u_t(x,0) = g(x),\ 0 \le x \le L,$$

$$u(0,t) = u(L,t) = 0,\ t \ge 0,$$

where $D^{\alpha}_{t} u(x,t)$ is Caputo fractional derivative [30], that is defined as follow

$$\frac{D^{\alpha} u(x,t)}{\partial t^{\alpha}} = D^{\alpha}_{t} u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-p)^{n-\alpha-1}} \frac{\partial^n u(p,x)}{\partial p^{\alpha}} dp, \ (n-1 < \alpha \le n), \quad (2)$$

if $\alpha = n \in N$, then we can write as:

$$D^{\alpha}_{t} u(x,t) = \frac{D^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{D^{n} u(x,t)}{\partial t^{n}}.$$

The numerical solutions for the suggested model (1) were derived by [31] using the Dufort Frankel difference scheme approach. Since the Caputo derivative [30] is the most frequently utilized derivative characteristic in the literature on fractional analysis, we adopted it in our model. The authors in [29] employed the explicit finite difference technique to solve the model (1). Additionally, they developed a first-order difference technique to investigate the stability analysis and obtained the approximate solutions of the given issue. To obtain the analytical solution to PHTPDE based on the Caputo fractional derivative of α order and $0 < \alpha \le 1$, this research

article implements an efficient analytical technique called MDLTM to solve the proposed model analytically, and this method is incredibly effective because it generates an accurate solution in relatively few steps than other approaches that need more steps.

This research paper is organized as follows: section 2 gives background information. Section 3 provides the general solution of fractional order PHTPDE defined by the Caputo derivative by the MDLTM. Section 4 presents the applications of MDLTM. Via section 5, the conclusion is given.

2 Basic concept

In this section, we define a few terms and give some fundamental details about the double Laplace Decomposition Method.

Definition 1. In the positive quadrant of the xt-plane, the function $u(x,t)$ as a function of two variables is defined. In [32], the double Laplace transform for the function $u(x,t)$ is specified as follows:

$$L_x L_t\{u(x,t)\} = \bar{u}(s,p) = \int_0^\infty \int_0^\infty e^{-px-st} u(x,t)\,dx\,dt \tag{3}$$

whenever the integral is present the numbers p and s are both complex numbers.

From this definition, we can thus deduce:

$$L_x L_t\{u(x)g(t)\} = \bar{u}(p)\bar{g}(s) = L_x\{u(x)\}L_t\{g(t)\} \tag{4}$$

Definition 2. The complex double integral formula is used to define the inverse double Laplace transform $L_x^{-1}L_t^{-1}\{\bar{u}(s,p)\} = u(x,t)$, as in [33]:

$$L_x^{-1}L_t^{-1}\{\bar{u}(s,p)\} = u(x,t) = \frac{1}{2i\pi}\left[\int_{c-i\infty}^{c+i\infty} e^{px}dp\right]\left[\int_{d-i\infty}^{d+i\infty} e^{st}\bar{u}(s,p)ds\right] \tag{5}$$

where c and d are appropriate real constants, and $\bar{u}(s,p)$ must be an analytical function for each p and s in the area denoted by the inequalities $\mathrm{Re}(p) \geq c$ and $\mathrm{Re}(s) \geq d$.

The double Laplace transform formula in [27] is used to transform the partial derivatives of any integer order.

$$L_x L_t\left[\frac{\partial^n u(x,t)}{\partial x^n}\right] = p^n\bar{u}(s,p) - \sum_{i=0}^{n-1} p^{n-1-i} L_t\left[\frac{\partial^i u(0,t)}{\partial x^i}\right]$$

$$L_x L_t\left[\frac{\partial^k u(x,t)}{\partial t^k}\right] = s^k\bar{u}(s,p) - \sum_{j=0}^{k-1} s^{n-1-i} L_x\left[\frac{\partial^j u(x,0)}{\partial x^j}\right]$$

$$L_x L_t\left[\frac{\partial^{k+n} u(x,t)}{\partial x^n \partial t^k}\right] =$$

$$p^n s^k\,\bar{u}(s,p) - \sum_{i=0}^{n-1} p^{n-1-i} L_t\left[\frac{\partial^i u(0,t)}{\partial x^i}\right] - \sum_{j=0}^{k-1} s^{n-1-j} L_x\left[\frac{\partial^j u(x,0)}{\partial x^j}\right] + \sum_{i=0}^{n-1}\sum_{j=0}^{k-1} p^{n-1-i} s^{n-1-j} L\left[\frac{\partial^{i+j} u(0,0)}{\partial x^{i+j}}\right]$$

Definition 3. According to [27], the double Laplace transform formula for Caputo fractional derivative (2) is defined as

$$L_x L_t\left[\frac{D^\alpha u(x,t)}{\partial t^\alpha}\right] = s^\alpha \bar{u}(s,p) - \sum_{k=0}^{n-1} s^{\alpha-1-k} L_x\left[\frac{D^k u(0,x)}{\partial t^k}\right]. \tag{6}$$

Definition 4. In [30], the gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad \mathrm{Re}(z) > 0.$$

(7)

3 The projected scheme

In order to solve fractional-order PHTE in the Caputo sense, this section develops the key concept of the double Laplace decomposition approach for the proposed problem.

For this purpose, we examine the fractional-order PHTE (1) in the following form

$$u_{tt}(x,t) = \lambda\, u_{txx}(x,t) - D_t^{\alpha} u(x,t) + u_{xx}(x,t) - u(x,t) + f(x,t), \tag{8}$$

which is $0 < x < L, t > 0$ and also subjected to the initial and boundary conditions

$$u(0,x) = h(x),\ u_t(x,0) = g(x), \tag{9}$$

where $0 \le x \le L$ and also

$$u(0,t) = u(L,t) = 0, \tag{10}$$

where $t \ge 0$. Here $\lambda > 0$ and $D^{\alpha}{}_t\mu(x,t)$ is Caputo fractional derivative of order $0 < \alpha \le 1$, while $f(x,t)$ is the given function. To get the general solution of equation (8), firstly we utilize double Laplace transform to equation (8), we obtain

$$s^2\bar{u}(s,p) - s\bar{u}(0,p) - \bar{u}_t(0,p) = L_xL_t\{\lambda\, u_{txx}(x,t) - D^{\alpha} u(x,t) + u_{xx}(x,t) - u(x,t)\} + \bar{f}(s,p), \tag{11}$$

where $\bar{f}(s,p) = L_xL_t\{f(x,t)\}$. After that, by applying a single Laplace transform to equations equation (9) and equation (10), we get

$$\begin{aligned} &\bar{u}(0,p) = \bar{h}(p),\\ &\bar{u}_t(0,p) = \bar{g}(p),\\ &\bar{u}(s,0) = \bar{u}(s,L) = 0,\ 0 \le x \le L,\ t \ge 0. \end{aligned} \tag{12}$$

by substituting formula (12) into formula (11) and simplifying, we obtain the following formula

$$\bar{u}(s,p) = \frac{1}{s^2} L_xL_t\{\lambda u_{txx}(x,t) - D^{\alpha}\mu(x,t) + u_{xx}(x,t) - u(x,t)\} + \bar{f}(s,p) + s\bar{h}(p) + \bar{g}(p)\ . \tag{13}$$

When we apply the inverse double Laplace transform for equation (13), yields

$$\begin{aligned} u(x,t) = {} & L_x^{-1}L_t^{-1}\frac{1}{s^2}\{L_xL_t\{\lambda u_{txx}(x,t) - D_t^{\alpha} u(x,t) + u_{xx}(x,t) - u(x,t)\}\}\\ & + L_x^{-1}L_t^{-1}\ \frac{1}{s^2}\bar{f}(s,p) + \frac{1}{s}\bar{h}(p) + \frac{1}{s^2}\bar{g}(p)\ . \end{aligned} \tag{14}$$

The general solutions to equation (13), can be expressed as an infinite series, as shown below

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{15}$$

Consequently, equation (13) becomes

$$\sum_{n=0}^{\infty} u_n(x,t) = L_x^{-1}L_t^{-1}\left(\frac{1}{s^2}L_xL_t\left(\lambda\frac{\partial^3}{\partial x^2\partial t}\sum_{n=0}^{\infty}u_n(x,t) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}u_n(x,t) - \frac{D^{\alpha}u(x,t)}{\partial t^{\alpha}}\sum_{n=0}^{\infty}u_n(x,t) - \sum_{n=0}^{\infty}u_n(x,t)\right)\right)$$
$$+ L_x^{-1}L_t^{-1}\ \frac{1}{s^2}\bar{f}(s,p) + \frac{1}{s}\bar{h}(p) + \frac{1}{s}\bar{g}(p)\ . \tag{16}$$

The general solution of equation (8) is described as the following formulas:

$$u_0(x,t) = L_x^{-1}L_t^{-1}\left(\frac{1}{s^2}\bar{f}(s,p) + \frac{1}{s}\bar{h}(p) + \frac{1}{s^2}\bar{g}(p)\right), \tag{17}$$

and

$$u_{n+1}(x,t) = L_x^{-1}L_t^{-1}\left(\frac{1}{s^2}L_xL_t\left(\lambda\frac{\partial^3}{\partial x^2\partial t}\sum_{n=0}^{\infty}u_n(x,t) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}u_n(x,t) - \frac{D^\alpha u(x,t)}{\partial t^\alpha}\sum_{n=0}^{\infty}u_n(x,t)\right)\right) - \sum_{n=0}^{\infty}u_n(x,t), n \geq 0. \tag{18}$$

We suppose that each part on the right side of (18) has an inverse double Laplace transform.

4 Applications of MDLTM

In this section, we provide an illustration and applicability of how to solve the fractional-order PHTE using the provided approach. We consider the following fractional-order PHTE [29], as

$$u_{tt}(x,t) = \lambda u_{txx}(x,t) - D_t^\alpha u(x,t) + u_{xx}(x,t) - u(x,t) + \left(2 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2\lambda t + 2\right)(t^2+1)\sin(x), \tag{19}$$

where $0 \leq x \leq \pi,\ 0 \leq t \leq 1, \lambda > 0,\ 0 < \alpha \leq 1$ subject to the initial and boundary conditions

$$u(x,0) = \sin(x),\ u_t(x,0) = 0,\ 0 \leq x \leq \pi \tag{20}$$

and also

$$u(0,t) = u(\pi,t) = 0,\ t \geq 0. \tag{21}$$

Taking the MDLTM to both sides of (19) and by using the initial conditions (20) and simplifying, we get

$$\begin{aligned}\bar{u}(s,p) &= L_xL_t\{u(x,t)\} \\ &= \frac{1}{s} + \frac{2}{s^3} + \frac{2}{s^{5-\alpha}} + \frac{2\lambda}{s^4} + \frac{4}{s^5} + \frac{2}{s^3}\left(\frac{1}{p^2+1}\right) \\ &\quad + \frac{1}{s^2}L_xL_t\{\lambda u_{txx}(x,t) - D_t^\alpha u(x,t) + u_{xx}(x,t) - u(x,t)\}.\end{aligned} \tag{22}$$

Equation (22), when transformed using the inverse double Laplace method, yields

$$\begin{aligned}u(x,t) &= \left(1 + 2t^2 + \frac{2}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{\lambda}{3}t^3 + \frac{1}{6}t^4\right)\sin(x) \\ &\quad + L_x^{-1}L_t^{-1}\left(\frac{1}{s^2}L_xL_t\{\lambda u_{txx}(x,t) - D_t^\alpha u(x,t) + u_{xx}(x,t) - u(x,t)\}\right).\end{aligned} \tag{23}$$

When the decomposition series is applied to equation (23) for $u(x,t)$, then we have

$$\begin{aligned}\sum_{n=0}^{\infty}u_n(x,t) &= \left(1 + 2t^2 + \frac{2}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{\lambda}{3}t^3 + \frac{1}{6}t^4\right)\sin(x) \\ &\quad + L_x^{-1}L_t^{-1}\left(\frac{1}{s^2}L_xL_t\left(\lambda\frac{\partial^3}{\partial x^2\partial t}\sum_{n=0}^{\infty}u_n(x,t) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}u_n(x,t) - \frac{D^\alpha u(x,t)}{\partial t^\alpha}\sum_{n=0}^{\infty}u_n(x,t)\right)\right) \\ &\quad - \sum_{n=0}^{\infty}u_n(x,t).\end{aligned} \tag{24}$$

The following recursive relationship can be deduced from equation (24)

$$u_0(x,t) = \left(1 + 2t^2 + \frac{2}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{\lambda}{3}t^3 + \frac{1}{6}t^4\right)\sin(x) \tag{25}$$

$$u_{n+1}(x,t) = L^{-1}L^{-1}\left(\frac{1}{s^2}L\left(L\left(\lambda\frac{\partial^3}{\partial x^2\partial t}\sum_{n=0}^{\infty}u_n(x,t) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}u_n(x,t) - \frac{D^\alpha u(x,t)}{\partial t^\alpha}\sum_{n=0}^{\infty}u_n(x,t)\right)\right)\right) - \sum_{n=0}^{\infty}u_n(x,t)\Bigg), n \geq 0. \tag{26}$$

From equation (26), we may get the following formula

$$u_1(x,t) = -\left(\frac{2}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{2}{\Gamma(7-2\alpha)}t^{6-2\alpha} + \frac{4\lambda}{\Gamma(6-\alpha)}t^{5-\alpha} + \frac{8}{\Gamma(7-\alpha)}t^{6-\alpha} + t^2 + \frac{1}{3}t^4 + \frac{\lambda}{30}t^5 + \frac{1}{90}t^6 + \frac{2\lambda}{3}t^3 + \frac{\lambda^2}{12}t^4 + \frac{\lambda}{10}t^5\right)\sin(x)$$

$$u_2(x,t) = \left(\frac{6\lambda}{\Gamma(6-\alpha)}t^{5-\alpha} + \frac{6\lambda}{\Gamma(8-2\alpha)}t^{7-2\alpha} + \frac{6\lambda^2}{\Gamma(7-\alpha)}t^{6-\alpha} + \frac{32\lambda}{\Gamma(8-\alpha)}t^{7-\alpha} + \frac{\lambda}{3}t^3 + \frac{16\lambda}{120}t^5 + \frac{\lambda^2}{36}t^6 + \frac{56\lambda}{\Gamma(8)}t^7 + \frac{\lambda^2}{6}t^4 + \frac{\lambda^3}{60}t^5 + \frac{12}{\Gamma(7-\alpha)}t^{6-\alpha} + \frac{12}{\Gamma(9-2\alpha)}t^{8-2\alpha} + \frac{24}{\Gamma(9-\alpha)}t^{8-\alpha} + \frac{1}{6}t^4 + \frac{16}{\Gamma(7)}t^6 + \frac{16}{\Gamma(9)}t^8 + \frac{2}{\Gamma(7-2\alpha)}t^{6-2\alpha} + \frac{2}{\Gamma(9-3\alpha)}t^{8-3\alpha}\right)\sin(x),$$

and so forth. The general solution of equation (26) can be expressed as an infinite series. As a result, the desired analytical solution is

$$u(x,t) = (t^2 + 1)\sin(x). \tag{27}$$

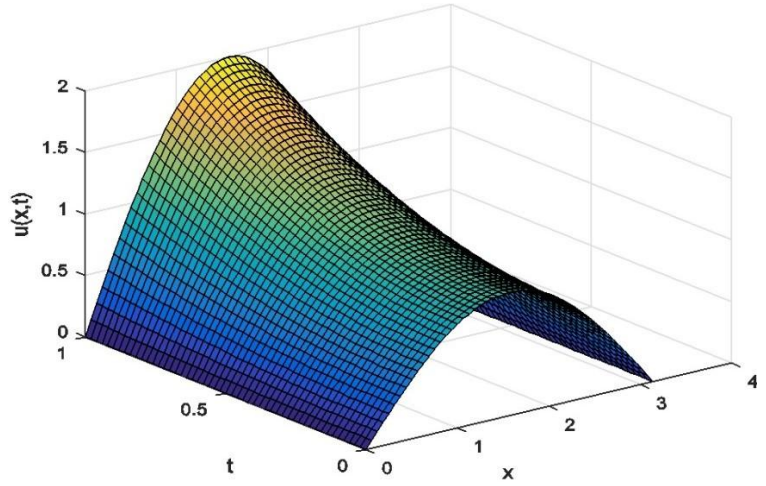


Fig. 1 3D simulation of the exact solution of equation (19) for $0 \leq x \leq \pi, 0 \leq t \leq 1, \lambda > 0$ and $0 \leq \alpha \leq 1$.

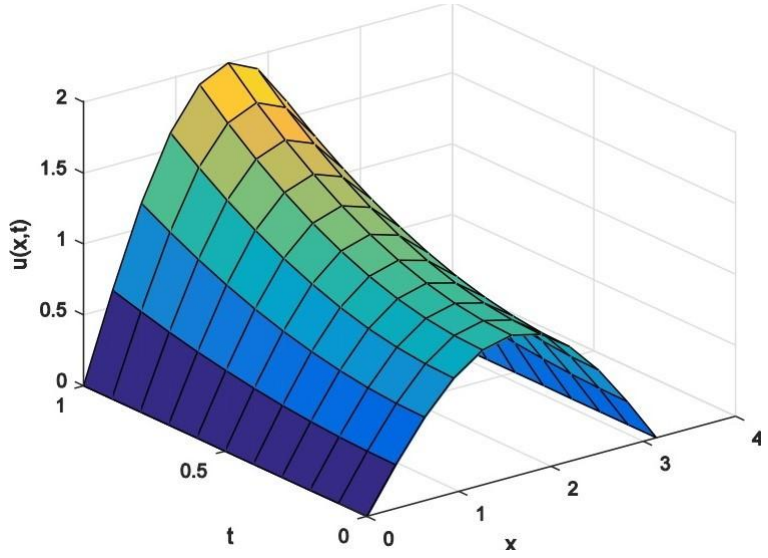


Fig. 2 3D simulation of the analytic solution of equation (19) for $0 \leq x \leq \pi, 0 \leq t \leq 1, \lambda > 0$ and $\alpha = 0.01$.

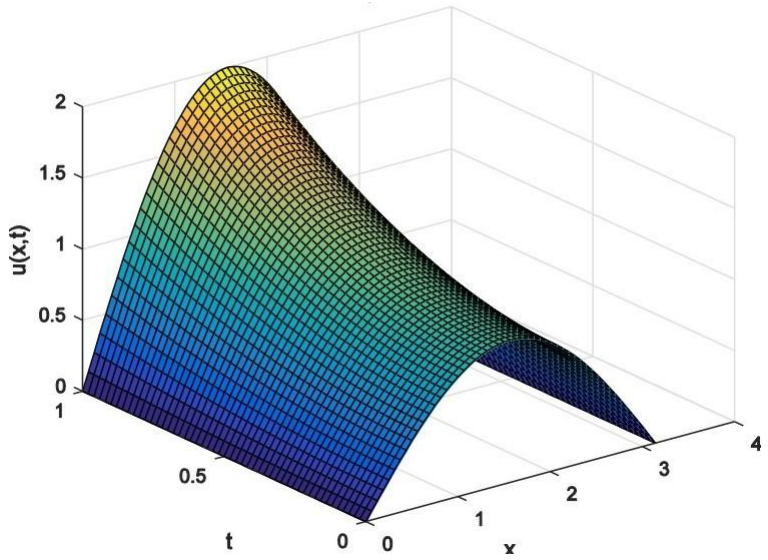


Fig. 3 3D simulation of the analytic solution of equation (19) for $0 \leq x \leq \pi, 0 \leq t \leq 1, \lambda > 0$ and $\alpha = 0.5$.

5 Conclusion

This investigation took into account the fractional order PHTPDE that is derived from the Caputo fractional derivative of order. Using the MDLT technique, we went over the suggested problem's general solution. A test case was provided as an example of an application. The analytical answer was obtained using the MDLT technique. The obtained result demonstrated the tremendous efficiency of this approach, as it only takes a few steps to construct an analytical solution, in contrast to other approaches that need more steps to accomplish the same goal. After discussing the modified double Laplace transform approach, this study offered a significant addition by suggesting an analytical solution to the fractional order pseudo-hyperbolic telegraph equation. A precise, closed-form expression based on mathematical functions or recognized processes is provided by the solution, which sheds light on the equation's behavior. Obtaining analytical solutions for fractional-order equations is notoriously difficult, therefore this accomplishment is all the more remarkable given the proposed technique.

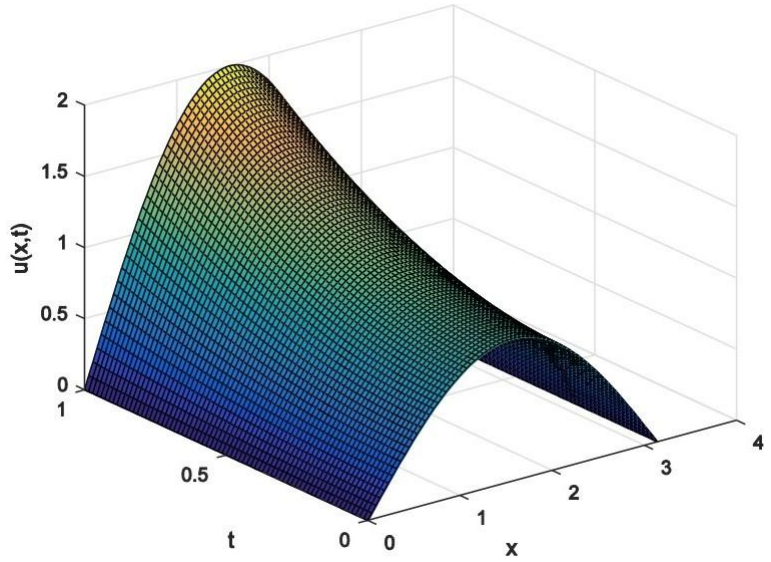


Fig. 4 3D simulation of the analytic solution of equation (19) for $0 \leq x \leq \pi, 0 \leq t \leq 1, \lambda > 0$ and $\alpha = 0.99$.

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