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Functions for Jacobi and Laguerre Polynomials of Multiple Variables with Double Generating

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Abstract: Finding some double generating functions for Jacobi and Laguerre polynomials in many variables is the goal of this work. We derive many double generating functions for Sister Celine's polynomials in two variables from these results. We also discuss many intriguing particular situations of our core discoveries.

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1. Introduction

The Jacobi polynomials of several variables $P_n^{(\alpha_1,\beta_1;\cdots;\alpha_r,\beta_r)}(x_1,\cdots,x_r)$ of Shrivastava [4] are defined by

$$P_n^{(\alpha_1,\beta_1;\cdots;\alpha_r,\beta_r)}(x_1,\cdots,x_r)=\frac{(1+\alpha_1)_n(1+\alpha_2)_n\cdots(1+\alpha_r)_n}{(n\,!)^r}$$

$$\times F^{1:1;\cdots;1}_{0:1;\cdots;1}\left[\begin{matrix}-n:1+\alpha_1+\beta_1+n\,;\cdots;\,1+\alpha_r+\beta_r+n\,;\\ -\;:\;\alpha_1+1\,;\cdots;\,\alpha_r+1\,;\end{matrix}\;\frac{1-x_1}{2}\,\cdots\,\frac{1-x_r}{2}\right], \tag{1}$$

where $(a)_n$ is the Pochhammer symbol defined as :

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & \text{if } n=0\\ a(a+1)(a+2)...(a+n-1), & \text{if } n=1,2,....\end{cases} \tag{2}$$

and $F^{p:\,q_1;\,...\,;q_n}_{l:\,m_1;\,...\,;m_n}[x_1,...,x_n]$ is the multivariable extension of the Kamp´e de F´eriet function (see Srivastava and Manocha [7]):

$$F\left[\begin{matrix}(a_p)\;:\;(b^{(1)}_{q_1})\;;...;\;(b^{(n)}_{q_n});\end{matrix}\;x_1,\;\ldots\;,x_n\right)=\sum_{S_1,\ldots,S_n=0}^{\infty}\Lambda(S_1,\ldots,S_n)\frac{x_1^{S_1}}{S_1!}\cdots\frac{x_n^{S_n}}{S_n!}$$

$$F^{p\,:\,q_1;\;\ldots\;;q_n}_{l\,:\,m_1;\;\ldots\;;m_n}\left(\begin{array}{l} \\ (c_l):\;(d^{(1)}_{m_1});\ldots;\;(d^{(n)}_{m_n}); \end{array}\right), \tag{3}$$

where

$$\Lambda(S_1,\ldots,S_n)=\frac{\prod\limits_{j=1}^{p}(a_j)_{S_1+\cdots+S_n}\prod\limits_{j=1}^{q_1}(b_j^{(1)})_{S_1}\cdots\prod\limits_{j=1}^{q_n}(b_j^{(n)})_{S_n}}{\prod\limits_{j=1}^{l}(c_j)_{S_1+\cdots+S_n}\prod\limits_{j=1}^{m_1}(d_j^{(1)})_{S_1}\cdots\prod\limits_{j=1}^{m_n}(d_j^{(n)})_{S_n}}. \tag{4}$$

and, for convergence of the multiple hypergeometric series in (3)

$$1+l+m_k-p-q_k\geq 0,\; k=1,\cdots,n;$$

the equality holds when, in addition, either

$$p>l \text{ and } \;|x_1|^{\frac{1}{p-l}}+\cdots+|x_n|^{\frac{1}{p-l}}<1;$$

or

$$p\leq l \text{ and } \max\{|x_1|,\cdots,|x_n|\}<1.$$

The Laguerre polynomials of several variables $L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ of Khan and Shukla [3] are defined by

$$L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)=\frac{\prod\limits_{j=1}^{r}(\alpha_j+1)_n}{(n!)^r}\,\Psi_2^{(r)}\left[-n;\;\alpha_1+1,\;\cdots,\;\alpha_r+1;\;x_1,\cdots,x_r\right] \tag{5}$$

where $\Psi_2^{(r)}$ is the confluent hypergeometric function of r-variables [7]

$$\Psi_2^{(r)}\;[a;\,c_1,\cdots,c_r;\,x_1,\cdots,x_r]=\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{(a)_{m_1+\cdots+m_r}}{(c_1)_{m_1}\cdots(c_r)_{m_r}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \tag{6}$$

The Appell's double hypergeometric function F_2 [7] is defined by

$$F_2\left[a,\,b,b';\,c,c';\,x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{7}$$

The Humbert's double hypergeometric function Ψ_1 [7] is defined by

$$\Psi_1\left[a,\,b;\,c,c';\;x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{8}$$

The triple hypergeometric functions ${}_3\Phi^{(1)}_A$ and ${}_3\Phi^{(2)}_A$ of Jain [2] are defined by

$${}_3\Phi_A^{(1)}\;[a,\,b_1,b_2;\,c_1,c_2,c_3;\,x,y,\,z]=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{9}$$

$${}_3\Phi_A^{(2)}\;[a,\,b_1;\,c_1,c_2,c_3;\,x,y,\,z]=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{10}$$

2. Double Generating Functions

In this section, we have proved the following double generating functions:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m\,!)^{r-1}\,(n\,!)^{s-1}\,(\lambda)_{m+n}}{\prod\limits_{j=1}^{r}(1+\alpha_j)_m\ \prod\limits_{j=1}^{s}(1+\gamma_j)_n}\times P_m^{(\alpha_1,\beta_1-m;\cdots;\alpha_r,\beta_r-m)}(x_1,\cdots,x_r)P_n^{(\gamma_1,\delta_1-n;\cdots;\gamma_s,\delta_s-n)}(y_1,\cdots,y_s)t^m(-t)^n$$

$$= F^{1:1;\cdots;1}_{0:1;\cdots;1}\left[\begin{array}{l}\lambda : 1+\alpha_1+\beta_1;\cdots;1+\alpha_r+\beta_r;1+\gamma_1+\delta_1;\cdots;1+\gamma_s+\delta_s;\\ - : \alpha_1+1;\cdots;\alpha_r+1;\gamma_1+1;\cdots;\gamma_s+1;\end{array}\ \frac{1}{2}(x_1-1)t,\cdots,\frac{1}{2}(x_r-1)t,\frac{1}{2}(1-y_1)t,\cdots,\frac{1}{2}(1-y_s)t\right], \tag{11}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\,(m\,!)^{r-1}\,(n\,!)^{s-1}}{\prod\limits_{j=1}^{r}(1+\alpha_j)_m\ \prod\limits_{j=1}^{s}(1+\beta_j)_n}L_m^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)\,L_n^{(\beta_1,\cdots,\beta_s)}(y_1,\cdots,y_s)t^m\,(-t)^n$$

$$= \Psi_2^{(r+s)}\,[\lambda; 1+\alpha_1,\cdots,1+\alpha_r, 1+\beta_1,\cdots,1+\beta_s; -x_1t,\cdots,-x_rt, y_1t,\cdots,y_st] \tag{12}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m\,!)^{r-1}\,(n\,!)^{s-1}(\lambda)_{m+n}}{\prod\limits_{j=1}^{r}(1+\alpha_j)_m\ \prod\limits_{j=1}^{s}(1+\gamma_j)_n}P_m^{(\alpha_1,\beta_1-m;\cdots;\alpha_r,\beta_r-m)}(x_1,\cdots,x_r)L_n^{(\gamma_1,\cdots,\gamma_s)}(y_1,\cdots,y_s)\,t^m(-t)^n$$

$$= F^{1:1;\cdots;1;0;\cdots;0}_{0:1;\cdots;1;1;\cdots;1}\left[\begin{array}{l}\lambda : 1+\alpha_1+\beta_1;\cdots;1+\alpha_r+\beta_r; - ;\cdots; - ;\\ - : \alpha_1+1;\cdots;\alpha_r+1;\gamma_1+1;\cdots;\gamma_s+1;\end{array}\ \frac{1}{2}(x_1-1)t,\cdots,\frac{1}{2}(x_r-1)t, y_1t,\cdots,y_st\right]. \tag{13}$$

Proof of (11): Denoting the left - hand side of (11) by S, expressing the two Jacobi polynomials of several variables as in

(1) and using certain well-known properties of Pochhammer symbol, we get

$$
\begin{aligned}
S &= \sum_{m=0}^{\infty}\sum_{p_1=0}^{m}\sum_{p_2=0}^{m-p_1}\cdots\sum_{p_r=0}^{m-p_1-\cdots-p_r}\frac{(\lambda)_m\,(-1)^{p_1+\cdots+p_r}}{(m-p_1-\cdots-p_r)\,!}\\
&\times\frac{(1+\alpha_1+\beta_1)_{p_1}\cdots(1+\alpha_r+\beta_r)_{p_r}}{(1+\alpha_1)_{p_1}\cdots(1+\alpha_r)_{p_r}\,p_1!\cdots p_r!}\,t^m\left(\frac{1-x_1}{2}\right)^{p_1}\cdots\left(\frac{1-x_r}{2}\right)^{p_r}\\
&\times\sum_{n=0}^{\infty}\sum_{q_1=0}^{n}\sum_{q_2=0}^{n-q_1}\cdots\sum_{q_s=0}^{n-q_1-\cdots-q_s}\frac{(\lambda+m)_n(-1)^{q_1+\cdots+q_s}}{(n-q_1-\cdots-q_s)\,!}\\
&\times\frac{(1+\gamma_1+\delta_1)_{q_1}\cdots(1+\gamma_s+\delta_s)_{q_s}}{(1+\gamma_1)_{q_1}\cdots(1+\gamma_s)_{q_s}\,q_1!\cdots q_s!}\,(-t)^n\left(\frac{1-y_1}{2}\right)^{q_1}\cdots\left(\frac{1-y_s}{2}\right)^{q_s}\\
&=\sum_{m=0}^{\infty}\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty}\cdots\sum_{p_r=0}^{\infty}\frac{(\lambda)_{m+p_1+\cdots+p_r}}{m\,!}\\
&\times\frac{(1+\alpha_1+\beta_1)_{p_1}\cdots(1+\alpha_r+\beta_r)_{p_r}}{(1+\alpha_1)_{p_1}\cdots(1+\alpha_r)_{p_r}\,p_1!\cdots p_r!}\,t^{m+p_1+\cdots+p_r}\left(\frac{x_1-1}{2}\right)^{p_1}\cdots\left(\frac{x_r-1}{2}\right)^{p_r}\\
&\times\sum_{n=0}^{\infty}\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\cdots\sum_{q_s=0}^{\infty}\frac{(\lambda+m+p_1+\cdots+p_r)_{n+q_1+\cdots+q_s}}{n\,!}\\
&\times\frac{(1+\gamma_1+\delta_1)_{q_1}\cdots(1+\gamma_s+\delta_s)_{q_s}}{(1+\gamma_1)_{q_1}\cdots(1+\gamma_s)_{q_s}\,q_1!\cdots q_s!}\,(-t)^{n+q_1+\cdots+q_s}\left(\frac{y_1-1}{2}\right)^{q_1}\cdots\left(\frac{y_s-1}{2}\right)^{q_s}\\
&=\sum_{p_1=0}^{\infty}\cdots\sum_{p_r=0}^{\infty}\sum_{q_1=0}^{\infty}\cdots\sum_{q_s=0}^{\infty}\frac{(\lambda)_{p_1+\cdots+p_r+q_1+\cdots+q_s}(1+\alpha_1+\beta_1)_{p_1}\cdots(1+\alpha_r+\beta_r)_{p_r}}{(1+\alpha_1)_{p_1}\cdots(1+\alpha_r)_{p_r}\;p_1!\cdots p_r!}\\
&\times\frac{(1+\gamma_1+\delta_1)_{q_1}\cdots(1+\gamma_s+\delta_s)_{q_s}}{(1+\gamma_1)_{q_1}\cdots(1+\gamma_s)_{q_s}\,q_1!\cdots q_s!}\left(\frac{(x_1-1)t}{2}\right)^{p_1}\cdots\left(\frac{(x_r-1)t}{2}\right)^{p_r}\left(\frac{(1-y_1)t}{2}\right)^{q_1}\cdots\left(\frac{(1-y_s)t}{2}\right)^{q_s}\\
&\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda+p_1+\cdots+p_r+q_1+\cdots+q_s)_{m+n}t^m(-t)^n}{m\,!\,n\,!}
\end{aligned}
$$

Since,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda+p_1+\cdots+p_r+q_1+\cdots+q_s)_{m+n}t^m(-t)^n}{m\,!\,n\,!}=1$$

the above series yields the right-hand side of (11). This completes the proof of (11). The results (12) and (13) can be proved by the similar manner. In view of the following relationship between the Sister Celine's polynomials of two variables $f_{m,n}$ [5] and Jacobi and Laguerre polynomials respectively:

$$f_{m,n}\left(\begin{matrix}(-m:1,1)\ :\ (1+\alpha+\beta,1),(\tfrac{1}{2},1),(1,1)\ ;\ (1+\gamma+\delta,1),(\tfrac{1}{2},1),(1,1)\\ -\ :\ (\alpha+1,1),(-m,1),(m+1,1)\ ;\ (\gamma+1:1),(-n,1),(n+1,1)\end{matrix}\ ;\ \frac{1-x}{2},\frac{1-y}{2}\right)$$

$$=\frac{(m\,!)^2P_m^{(\alpha,\,\beta-m;\gamma,\delta-m)}(x,y)}{(\alpha+1)_m\,(\gamma+1)_m}, \tag{14}$$

$$f_{m,n}\left(\begin{matrix}-\ :\ (1+\alpha+\beta,1),(\tfrac{1}{2},1),(1,1)\ ;\ (1+\gamma+\delta,1),(\tfrac{1}{2},1),(1,1)\\ -\ :\ (\alpha+1,1),(m+1,1)\ ;\ (\gamma+1:1),(n+1,1)\end{matrix}\ ;\ \frac{1-x}{2},\frac{1-y}{2}\right)$$

$$=\frac{m\,!\,n\,!\,P_m^{(\alpha,\,\beta-m)}(x)P_n^{(\gamma,\,\delta-n)}(y)}{(\alpha+1)_m\,(\gamma+1)_n}, \tag{15}$$

$$f_{m,n}\left(\begin{matrix}(-m:1,1)\ :\ (\tfrac{1}{2},1),(1,1)\ ;\ \tfrac{1}{2},1),(1,1)\\ -\ :\ (-m,1),(m+1,1),(1+\alpha:1)\ ;\ (-n,1),(n+1,1),(1+\beta,1)\end{matrix}\ ;\ x,y\right)$$

$$= \frac{(m\,!)^2 L_m^{(\alpha,\,\beta)}(x,y)}{(1+\alpha)_m\,(1+\beta)_m} \tag{16}$$

and

$$f_{m,n}\left(\begin{matrix} - : & (\frac{1}{2},1)\,,(1,1) & ; & \frac{1}{2},1)\,,(1,1) & ; \\ - : & (m+1,1),(1+\alpha,\,1) & & & \end{matrix}\; x\,,y\right) = \frac{m\,!\,n\,!\,L_m^{(\alpha)}(x)L_n^{(\beta)}(y)}{}\,, \tag{17}$$

$; (n+1,1),(1+\beta,1)\,;$ $(1+\alpha)^{m\,(1+\beta)n}$

we get from (11), (12) and (13) respectively the following results :

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m\,!\,n\,!}$$

$; (1+\alpha_2+\beta_2\,,1)\,,(\frac{1}{2},1)\,,(1\,,1)$ $\quad 1-x_1 \quad 1-x_2$

$- \quad : (\alpha_1+1,1),(-m,1),(m+1,1) \quad ; (\alpha_2+1:1),(-n,1),(n+1,1) \qquad 2 \qquad 2$

$f_{m,n}\left((-m:1,1) \;:\; (1+\alpha_1+\beta_1\,,1)\,,(\frac{1}{2},1)\,,(1\,,1) \right.$ $\quad ; \quad ,$

$; \quad (1+\gamma_2+\delta_2\,,1)\,,(\frac{1}{2},1)\,,(1\,,1) \quad \frac{1-y_1}{2} \quad \frac{1-y_2}{2}$

$- \quad : \quad (\gamma_1+1,1),(-n,1),(n+1,1) \quad ; (\gamma_2+1:1),(-m,1),(m+1,1)$

$1:1;1;1;1 \quad \lambda \quad : 1+\alpha_1+\beta_1\,;\,1+\alpha_2+\beta_2\,;\,1+\gamma_1+\delta_1\,;\,1+\gamma_2+\delta_2\,;$

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$= \quad 0:1;1;1;1 \quad -: \quad \alpha_1+1 \quad ; \quad \alpha_2+1 \quad ; \quad \gamma_1+1 \quad ; \quad \gamma_2+1 \quad ;$

$\times f_{n,m}\left((-n:1,1) \;:\; (1+\gamma_1+\delta_1\,,1)\,,(\frac{1}{2},1)\,,(1\,,1) \right.$ $\quad ; \quad ,$

$$\left.\frac{1}{2}(x_1-1)t,\,\frac{1}{2}(x_2-1)t\,,\,\frac{1}{2}(1-y_1)t,\,\frac{1}{2}(1-y_2)t\right] \tag{18}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m\,!\,n\,!}f_{m,n}\left(\begin{matrix} - : & (1+\alpha+\beta\,,1)\,,(\frac{1}{2},1)\,,(1\,,1) & (1+\gamma+\delta\,,1)\,,(\frac{1}{2},1)\,,(1\,,1) \\ - : & (\alpha+1\,,1)\,,(m+1\,,1) & (\gamma+1\,:\,1)\,,(n+1\,,1) \end{matrix}\;;\,\frac{1-x}{2}\,,\,\frac{1-y}{2}\right)$$

;

;

$$= F_2\left[\lambda\,,1+\alpha+\beta,\,1+\gamma+\delta\,;\,\alpha+1\,,\gamma+1\,;\;\tfrac{1}{2}(x-1)t\,,\;\tfrac{1}{2}(1-y)t\right] \tag{19}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m\,!\,n\,!}$$

$$f_{m,n}\left(\begin{matrix} (-m:1,1) : & (\frac{1}{2},1)\,,(1\,,1) & ; & (\frac{1}{2},1)\,,(1\,,1) & ; \\ & & & (-n,1)\,,(n+1\,,1),(\alpha_2+1,\,1) & ; \end{matrix}\; x_1\,,x_2\right)$$

$- \quad : (-m,1),(m+1,1),(\alpha_1+1,1)\,;$

$$\times f_{n,m}\left(\begin{matrix} (-n:1,1) : & (\frac{1}{2},1)\,,(1\,,1) & ; & (\frac{1}{2},1)\,,(1\,,1) & ; \\ & & & (-m,1)\,,(m+1\,,1),(\beta_2+1,\,1) & ; \end{matrix}\; y_1\,,y_2\right)$$

$- \quad : (-n,1),(n+1,1),(\beta_1+1,1)\,;$

$$= \Psi_2^{(4)}\left[\lambda; 1+\alpha_1, 1+\alpha_2, 1+\beta_1, 1+\beta_2; -x_1 t, -x_2 t, y_1 t, y_2 t\right] \tag{20}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m!\,n!} f_{m,n}\left(\begin{matrix} - : & \left(\tfrac{1}{2},1\right),(1,1) \\ - : & (m+1,1),(\alpha+1,1)\end{matrix}\;;\; \begin{matrix}\left(\tfrac{1}{2},1\right),(1,1) \\ (n+1,1),(\beta+1,1)\end{matrix}\;;\; x, y\right)$$

$$= \Psi_2(\lambda; \alpha+1, \beta+1; -xt, yt) \tag{21}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m!\,n!}$$

$$f_{m,n}\left(\begin{matrix}(-m:1,1) : & (1+\alpha_1+\beta_1,1),\left(\tfrac{1}{2},1\right),(1,1) & ; & (1+\alpha_2+\beta_2,1),\left(\tfrac{1}{2},1\right),(1,1) \\ - & : (\alpha_1+1,1),(-m,1),(m+1,1) & ; & (\alpha_2+1:1),(-n,1),(n+1,1)\end{matrix}\;;\; \frac{1-x_1}{2}, \frac{1-x_2}{2}\right)$$

$$\times f_{n,m}\left(\begin{matrix}(-n:1,1) : & \left(\tfrac{1}{2},1\right),(1,1) & ; & \left(\tfrac{1}{2},1\right),(1,1) \\ - & : (-n,1),(n+1,1),(\gamma_1+1,1) & ; & (-m,1),(m+1,1),(\gamma_2+1,1)\end{matrix}\;;\; y_1, y_2\right)$$

$$= F^{1:1;1;0;0}_{0:1;1;1;1}\left[\begin{matrix}\lambda : & 1+\alpha_1+\beta_1; & 1+\alpha_2+\beta_2; & -; & -; \\ - : & \alpha_1+1; & \alpha_2+1; & \gamma_1+1; & \gamma_2+1;\end{matrix}\;\tfrac{1}{2}(x_1-1)t, \tfrac{1}{2}(x_2-1)t, y_1 t, y_2 t\right] \tag{22}$$

3. Special Cases

In (11) putting $r = s = 2$, $x_1 = 1 - x$, $x_2 = x + 1$, $y_1 = 1 - y$, $y_2 = y + 1$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, $\gamma_1 = \gamma_2 = \gamma$, $\delta_1 = \delta_2 = \delta$ and using the result [6]

$$F_2\left[\alpha, \beta, \beta; \gamma, \gamma; x, -x\right] = {}_4F_3\left[\tfrac{1}{2}\alpha, \tfrac{1}{2}\alpha+\tfrac{1}{2}, \beta, \gamma-\beta; \gamma, \tfrac{1}{2}\gamma, \tfrac{1}{2}\gamma+\tfrac{1}{2}; x^2\right], \tag{23}$$

we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m!\,n!\,(\lambda)_{m+n}\,t^m(-t)^n P_m^{(\alpha,\beta-m;\,\alpha,\beta-m)}(1-x, x+1)P_n^{(\gamma,\delta-n;\gamma,\delta-n)}(1-y, y+1)}{(\alpha+1)_m(\alpha+1)_m(\gamma+1)_n(\gamma+1)_n}$$

$$= F^{2\,:\,2\,;\,2}_{0\,:\,3\,;\,3}\left[\begin{matrix}\tfrac{1}{2}\lambda, \tfrac{1}{2}\lambda+\tfrac{1}{2} : & 1+\alpha+\beta, -\beta & ; & 1+\gamma+\delta, -\delta & ; \\ - : & \alpha+1, \tfrac{1}{2}(\alpha+1), \tfrac{1}{2}(\alpha+2) & ; & \gamma+1, \tfrac{1}{2}(\gamma+1), \tfrac{1}{2}(\gamma+2) & ;\end{matrix}\;\left(\frac{xt}{2}\right)^2, \left(\frac{yt}{2}\right)^2\right] \tag{24}$$

Similarly, in (11) putting $r = s = 3$, $x_1 = y_1 = x$, $x_2 = y_2 = y$, $x_3 = y_3 = z$, $\alpha_1 = \gamma_1 = \alpha$, $\alpha_2 = \gamma_2 = \gamma$, $\alpha_3 = \gamma_3 = \varepsilon$, $\beta_1 = \delta_1 = \beta$, $\beta_2 = \delta_2 = \delta$, $\beta_3 = \delta_3 = \nu$ and using the result (23), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m!\,n!\,(\lambda)_{m+n}\,t^m(-t)^n P_m^{(\alpha,\beta-m;\,\gamma,\delta-m;\,\varepsilon,\nu-m)}(x,y,z)P_n^{(\alpha,\beta-n;\,\gamma,\delta-n;\,\varepsilon,\nu-n)}(x,y,z)}{(\alpha+1)_m(\gamma+1)_m(\varepsilon+1)_m(\alpha+1)_n(\gamma+1)_n(\varepsilon+1)_n}$$

$$= F^{2:2;2;2}_{0:3;3;3}\left[\begin{matrix}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : 1+\alpha+\beta, -\beta ; 1+\gamma+\delta, -\delta ; 1+\varepsilon+\nu, -\nu ; \\ - : \alpha+1, \frac{1}{2}(\alpha+1), \frac{1}{2}(\alpha+2) ; \gamma+1, \frac{1}{2}(\gamma+1), \frac{1}{2}(\gamma+2) ; \varepsilon+1, \frac{1}{2}(\varepsilon+1), \frac{1}{2}(\varepsilon+2) ;\end{matrix} \left(\frac{(x-1)t}{2}\right)^2, \left(\frac{(y-1)t}{2}\right)^2, \left(\frac{(z-1)t}{2}\right)^2\right] \tag{25}$$

In (12) putting $r = s = 2$, $x_1 = x_2 = x$, $y_1 = y_2 = y$ and using the result [6]

$$\Psi_2(\alpha; \gamma, \gamma'; x, x) = {}_3F_3\left[\begin{matrix}\alpha, \frac{1}{2}(\gamma+\gamma'), \frac{1}{2}(\gamma+\gamma'-1) ; \\ \gamma, \gamma', \gamma+\gamma'-1 ;\end{matrix} 4x\right], \tag{26}$$

we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\lambda)_{m+n}\, m!\, n!\, t^m\, (-t)^n}{(\alpha_1+1)_m\, (\alpha_2+1)_m\, (\beta_1+1)_n\, (\beta_2+1)_n} L_m^{(\alpha_1,\alpha_2)}(x, x)\, L_n^{(\beta_1,\beta_2)}(y, y)$$
$$= F^{1:2;2}_{0:3;3}\left[\begin{matrix}\lambda : \frac{1}{2}(\alpha_1+\alpha_2+1), \frac{1}{2}(\alpha_1+\alpha_2+2) ; \frac{1}{2}(\beta_1+\beta_2+1), \frac{1}{2}(\beta_1+\beta_2+2) ; \\ - : \alpha_1+1, \alpha_2+1, \alpha_1+\alpha_2+1 ; \beta_1+1, \beta_2+1, \beta_1+\beta_2+1 ;\end{matrix} -4xt, 4yt\right] \tag{27}$$

Similarly,in (12) putting $r = s = 2$, $x_1 = x$, $x_2 = -x$, $y_1 = y$, $y_2 = -y$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and using the result [6]

$$\Psi_2(\alpha; \gamma, \gamma; x, -x) = {}_2F_3\left[\begin{matrix}\frac{1}{2}\alpha, \frac{1}{2}(\alpha+1) ; \\ \gamma, \frac{1}{2}\gamma, \frac{1}{2}(\gamma+1) ;\end{matrix} x^2\right], \tag{28}$$

we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\lambda)_{m+n}\, m!\, n!\, t^m\, (-t)^n}{(\alpha+1)_m\, (\alpha+1)_m\, (\beta+1)_n\, (\beta+1)_n} L_m^{(\alpha,\alpha)}(x, -x)\, L_n^{(\beta,\beta)}(y, -y)$$
$$= F^{2:0;0}_{0:3;3}\left[\begin{matrix}\frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2} : - ; - ; \\ - : \alpha+1, \frac{1}{2}(\alpha+1), \frac{1}{2}(\alpha+2) ; \beta+1, \frac{1}{2}(\beta+1), \frac{1}{2}(\beta+2) ;\end{matrix} -x^2, -y^2\right] \tag{29}$$

In (12) putting $r = s = 1$, $\alpha_1 = \beta_1 = \alpha$, $\lambda = \alpha + 1$ and using the result [6]

$$\Psi_2(\gamma; \gamma, \gamma; x, y) = e^{x+y}\, {}_0F_1(-; \gamma; xy), \tag{30}$$

we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, t^m\, (-t)^n}{(\alpha+1)_m\, (\alpha+1)_n} L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(y) = \exp(ty - tx)\, {}_0F_1(-; \alpha+1; -t^2xy), \tag{31}$$

which is well known generating function of Exton [1].

In (13) putting ($r = 1$, $s = 1$), ($r = 2$, $s = 1$) and ($r = 1$, $s = 2$), we get respectively the following results:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\ t^m\ (-t)^n}{(\alpha+1)_m\,(\gamma+1)_n}P_m^{(\alpha,\beta-m)}\,(x)\ L_n^{(\gamma)}\,(y)=\ \Psi_1\ \left[\lambda\,,\,1+\alpha+\beta;\,\alpha+1\,,\,\gamma+1\,;\ \ \tfrac{1}{2}\,(x-1)t\,,\ yt\right]$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\ t^m\ (-t)^n}{(\alpha_1+1)_m(\alpha_2+1)_m\,(\gamma_1+1)_n}P_m^{(\alpha_1,\beta_1-m\ ;\,\alpha_2,\beta_2-m)}\,(x_1,\ x_2)\,L_n^{(\gamma_1)}\,(y_1) \quad (32)$$

$$={}_3\Phi_A^{(1)}\ \left[\lambda\,,\,1+\alpha_1+\beta_1,1+\alpha_2+\beta_2;\,1+\alpha_1,\,1+\alpha_2\,,1+\gamma_1\,;\ \ \tfrac{1}{2}\,(x_1-1)t,\tfrac{1}{2}\,(x_2-1)t,\ \ y_1t\right] \quad (33)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\ t^m\ (-t)^n}{(\alpha_1+1)_m(\gamma_1+1)_n\,(\gamma_2+1)_n}P_m^{(\alpha_1,\beta_1-m\ )}\,(x_1)\,L_n^{(\gamma_1,\gamma_2)}\,(y_1,y_2)$$

$$={}_3\Phi_A^{(2)}\ \left[\lambda\,,\,1+\alpha_1+\beta_1;\,1+\alpha_1,\,1+\gamma_1\,,1+\gamma_2\,;\ \ \tfrac{1}{2}\,(x_1-1)t,y_1t,\ \ y_2t\right]. \quad (34)$$

References

[1] [1] H. Exton, An extension of a Bailey theorem suggests a numerous generating functions, Pure Appl. Math. Sci., 37(1993), 1–12.[2] R.N. Jain, The confluent hypergeometric functions of three variables, Publications of the National Academy of Sciences, 36(1966), 395–408.
[3] On Laguerre polynomials in numerous variables, M.A. Khan and A.K. Shukla, Bull. Cal. Math. Soc., 89(1997), 155-164. Regarding Jacobi polynomials in many variables, H.S.P. Shrivastava, Integral Trans. Spec. Funct., 10(2000), 61-70.
[5] On Sister Celine's polynomials of many variables, H.S.P. Shrivastava, Math. And Comp. Appl., 9(2004), 309-320. [6] Multiple Gaussian Hypergeometric Series, by H.M. Srivastava and P.W. Karlsson, Halsted Press, New York, 1985.
[7] A Treatise on Generating Functions by H.M. Srivastava and H.L. Manocha, Halsted Press,