International Journal of Mathematical Modelling and Computation
Vol 2, Issue 1 2026

Categorization of mathematical models of erratic water flow in irrigation systems' major canals

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Article Info
Received:22-11-2025 Revised:05-12-2025 Accepted:07-01-2026 Published:17-01-2026

ABSTRACT: It is now feasible to model and manage water resources in all water management systems and facilities, including river sections, reservoirs, and canals (gravity and with machine water lifting systems), thanks to the advancements in computerization throughout the world and in recent years in the Central Asian states. You can use a computer to model the behavior of individual water facilities and how they interact, accounting for all the elements that affect their operation under settings that are similar to real-world ones. As a result, this article offers an analytical evaluation of recent research on mathematical models of unstable water movement in irrigation systems' main canals, along with a discussion of their benefits and drawbacks.

KEY WORDS: mathematical model, unsteady flow of water, main canals, optimal control problems, fundamental solution, differential equations, hydraulic structures.

I.INTRODUCTION

Only quantitative and qualitative changes in the characteristics of water management systems and facilities may be acquired through mathematical modeling techniques due to their wide spatial expanse and numerous technological and technical elements.
The selection of mathematical models that will accurately describe the necessary processes of water supply and distribution in water systems and facilities is a very difficult task because there is currently no unified systematic approach to modeling the dynamics of water facilities, only a broad class of mathematical models of individual objects of varying degrees of complexity.

Static models, dynamic models with lumped parameters, and dynamic models with distributed parameters are the three categories into which the primary models used to represent the control of irrigation systems' canals can be separated (Fig. 1). The beginning assumptions, the quantity and kind of primary data, and the level of specificity of the outcomes are all different among these models.[1, 2, 3, 4]

Different algebraic relationships between searching and known parameters are used in static models. These models include hydrological series models, several versions of regression models, etc. These models' parameters are established utilizing identification techniques (such as maximum likelihood and least squares methods) based on observational data and large-scale tests.[5]

Static models have the following benefits: they are easy to use, require little initial data, can be used with computers, and provide results quickly. With their assistance, the features of the watercourse in the areas where some observations were made can be determined, with a certain degree of likelihood.
The limited ability to apply static models to created things is one of its drawbacks. Outside of the boundaries and ranges of observations, models are not very useful and offer little insight into the process.

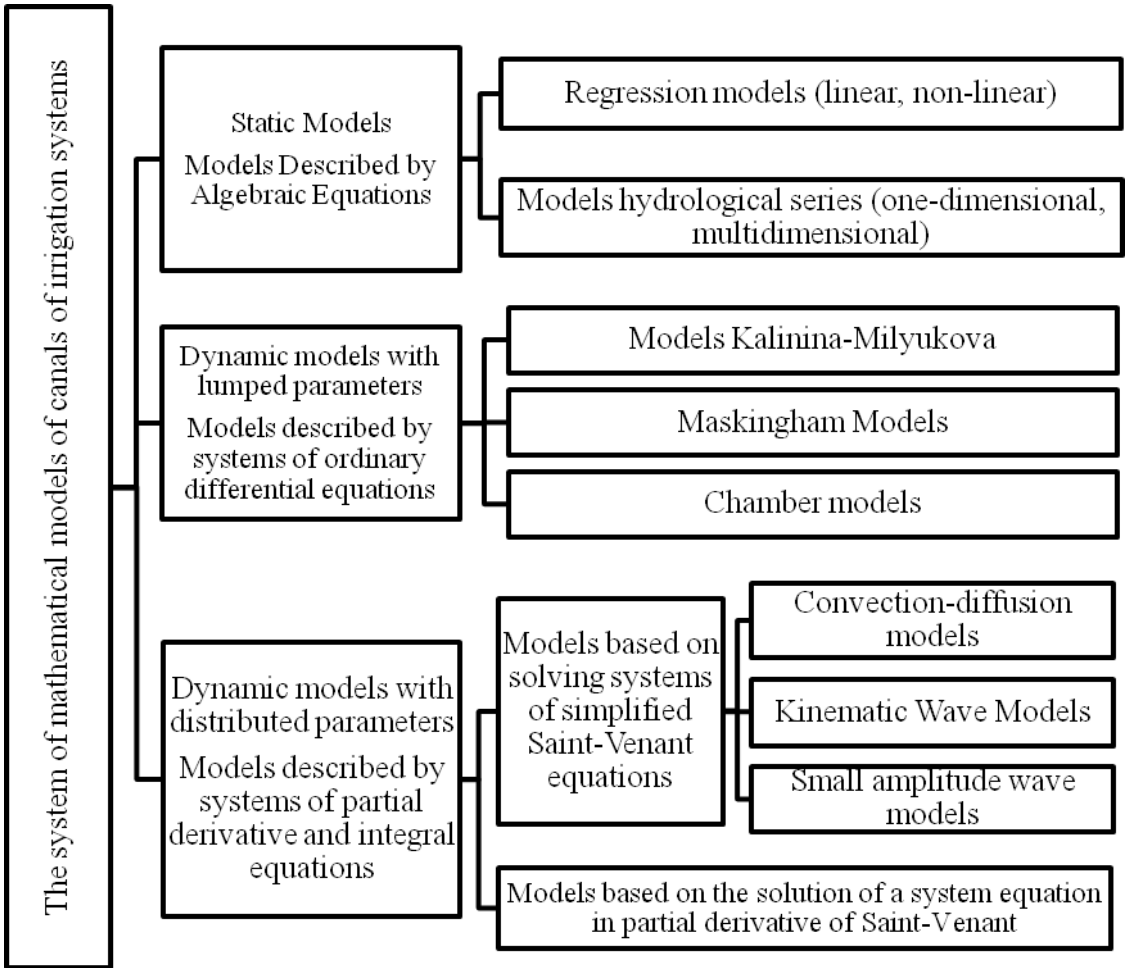


Fig.1. The system of mathematical models of canals of irrigation systems

II. METHODS AND RESULTS

Dynamic models with lumped parameters. Models with lumped parameters are based on the assumption that the length of the considered section is small compared to the wavelength. In these models, the size of the site is neglected and it is believed that its characteristics are concentrated at one point. Using these models, the characteristics of the water flow are calculated only in individual sections of the site. Models with lumped parameters also include hydrological models, such as linear and nonlinear reservoir models, Kalinin-Milyukov models, Maskingham model, etc. [1, 2, 3, 4].

The essence of the Kalinin-Milyukov model is the assumption that reach of riverbed has a length L (characteristic site) has the same regulatory effect on the release wave, as well as the reservoir equal in volume to it. In other words, with unsteady water movement, the uniqueness of the dependence of the volume of water in the area on the flow rate Q in its closing range: $W = f(Q_y)$

The length of the characteristic profile is determined through the parameters of steady motion - flow rate Q_y, slope $-i_y$ and the ordinate of the free surface Z_y

$$L = \frac{Q_y}{i_y}\frac{dZ_y}{dQ_y} \tag{1}$$

At the same time, it is assumed that the site can be divided into n characteristic sites in which there is an unambiguous relationship between the volume of water and the flow rate in the closing sections. Additionally, assuming that this relationship is linear, the calculation of the hydrograph can be reduced to solving a system of equations

$$\left.\begin{aligned} \frac{dQ_1}{dt} &= \frac{1}{\sigma_1}(q - Q_1), \\ \frac{dQ_2}{dt} &= \frac{1}{\sigma_2}(q - Q_2), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dQ_n}{dt} &= \frac{1}{\sigma_n}(q - Q_n), \end{aligned}\right\} \tag{2}$$

where q - inlet flow rate; Q_i - flow rate in the closing range i-th characteristic site; τ_i - coefficient of proportionality between the volume of water and the flow rate in the closing range i-th section (run-time on a characteristic section).

This system can be solved by any operator method.

As a result of solving this system, the dependence of water flow on time and riverbed parameters is obtained, which can be used for the calculation of control actions.

The Maskingham model is based on the solution of the water balance equation of the following form

$$\Delta t\frac{\left(W_н^t - W_к^t\right)}{} + \Delta t\frac{\left(W_н^{t+1} - W_к^{t+1}\right)}{} = K\left[\lambda\left(W_н^{t+1} - W_н^t\right) + (1-\lambda)\left(W_к^{t+1} - W_к^t\right)\right], \tag{3}$$

where $W_н^t$ and $W_н^{t+1}$ - volume of water in the initial section at a time t and $t+1$;

Δt - estimated time interval between t and $t+1$;

K – coefficient depending on the hydraulic parameters of the site for prismatic riverbeds with a constant slope $K = \frac{Cl}{\sqrt{i}}$; l – the site length.

Solving equation (3) with respect to W_K^{t+1}, we get

$$W_K^{t+1} = \frac{\left(\Delta t/2 + K\lambda\right)W^t - \left[\left(\Delta t/2 - K(1-\lambda)\right)\right]W^t + \left[\Delta t/2 - K\lambda\right]W^{t+1}}{\Delta t/2 + K(1-\lambda)} \tag{4}$$

or

$$W_{к}^{t+1} = C_1 W_{н}^{t+1} + C_2 W_{н}^{t} + C_3 W_{к}^{t}. \quad (5)$$

Here

$$C_1 = \frac{\Delta t/2 - K\lambda}{\Delta t/2 + K(1-\lambda)}, C_2 = \frac{\Delta t/2 + K\lambda}{\Delta t/2 + K(1-\lambda)}, C_3 = -\frac{\Delta t/2 - K(1-\lambda)}{\Delta t/2 + K(1-\lambda)}.$$

Selection K and Δ - the main problem for this method. They can be determined from observational data using equations (4) - (5).

This technique can be utilized for management purposes and enables you to determine the relationship between the volume of water in the riverbed and the flow rate. This approach has been modified by Kunzh and Paps [5]. Models were employed in [2, 5] to compute the erratic flow of water in river segments. River segments are split up into several chambers in these models, and a balance differential equation is created for each chamber. The aforementioned dynamic models with lumped parameters have the following advantages: they are simpler, require less initial data, require comparatively few computations, are deeper than stochastic models, capture the essence of the process being described, provide a higher level of detail in the calculation results, and can be calculated and applied in real time. The drawbacks of models with lumped parameters include the requirement for meticulous justification in each unique application instance due to the usage of assumptions that are not always clear-cut and do not follow from any more general rules.

The Saint-Venant equation primarily describes dynamic models with distributed parameters, which are predicated on the following assumptions and account for the distribution of flow motion parameters in river and canal sites over the length and time:

• The flow is one-dimensional, meaning that the water level is horizontal in the transverse direction and the cross-sectional speed is the same; • The pressure is hydrostatic because the streamlines' curvature is minor and their vertical accelerations are minimal;

• The equations of resistance, which are comparable to those used in the steady-state flow of water, can account for the influence of friction at the borders and turbulence of the flow;

• The cosine of the angle formed by the bottom line and the horizontal can be taken to equal unity, and the average slope of the riverbed bottom is small.

We see that the forms of flow under study are realized in riverbeds, whose cross sections can vary along the axis and have any shape, but this change is constrained by the need that the flow lines' curvature be minimal. The mass conservation law is the integral form of the Saint-Venant equations [1,26,27]:

$$\int_{x_1}^{x_2} (\omega_{t_2} - \omega_{t_1})\, dx + \int_{t_1}^{t_2} (Q_{x_2} - Q_{x_1})\, dt = 0 \quad (6)$$

and momentum conservation

$$\int_{x_1}^{x_2}\left(Q_{t_2}-Q_{t_1}\right)dx=\int_{t_1}^{t_2}\left[\left(\upsilon^2\omega\right)_{x_1}-\left(\upsilon^2\omega\right)_{x_2}\right]dt+$$
$$+g\int_{t_1}^{t_2}\left(I_{x_1}^{1}+I_{x_2}^{2}\right)dt-g\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\rho I^2+\omega\left(I_0-I_f\right)\right]dxdt, \quad (7)$$

where $Q=\upsilon\omega$;

$$I^1=\int_0^{h(x)}\left[h(x)-\eta\right]\sigma(x,\eta)d\eta$$
$$I^2=\int_0^{h(x)}\left[\left(\right)\eta\right]\left[\frac{\partial\sigma}{}\right]\eta;$$
$$I=\int_0^{}\left[h\,x-\left|\frac{}{\partial x}\right|_{h=h_0}\right]d$$

$0\qquad\partial Z_q\qquad f$

$I=-\frac{}{\partial x}$; I – friction slope; $\sigma(x,h)$- cross section width; moreover $\sigma(x,h)=\sigma(x)$ - top flow width; ω - living area; υ – uniform speed across the cross section; x – coordinate along the length of the site; t – time; h - water level; Z_q – bottom mark.

The differential form of the system of one-dimensional Saint-Venant equations has the form

$$i-\frac{\partial h}{\partial x}=\frac{v}{g}\frac{\partial v}{\partial x}+\frac{1}{g}\frac{\partial v}{\partial t}+\frac{|v|v}{C^2R}+\frac{qv}{g\omega}, \quad (8)$$

$$\frac{\partial\omega}{\partial t}+\frac{\partial Q}{\partial x}=q;$$

where C – Shezi coefficient; R – hydraulic radius; q – lateral inflow.

The aforementioned models have the advantage of using a limited set of widely recognized and extensively tested initial positions, as well as a precise and rigorous mathematical definition of the emergent difficulties.

The adaptability of hydraulic versions is a major benefit. They are useful for both designing and managing portions of rivers and canals. Hydraulic models nearly usually enable the interpolation and extrapolation of process characteristics over a broad range, as well as the acquisition of process characteristics with the necessary detail and acceptable inaccuracy. As a result, despite their high cost of production, they are widely used. This mainly relates to mathematical models that are one-dimensional [6, 7, 8].

The processes in riverbeds where the so-called non-transit zones—shrubbed or other areas of the river where the water barely moves—emerge are primarily responsible for the drawbacks of hydraulic models. Non-transit zones should not be considered in the live portion of the flow since they serve as storage capacity. The widely used one-dimensional equations of water transport do not account for transit zones since methods for doing so have not yet been devised. The mentioned drawbacks of hydraulic types are negligible because, with adequate maintenance, non-transit zones hardly ever arise in canals.

For the study of dynamic processes in water facilities and systems, hydraulic models are therefore of the highest interest.

The models that have been shown can be categorized based on the methods of solution. Three categories can be used to classify current approaches to solving Saint-Venant equations. The first comprises solutions derived from attempts to utilize strict mathematical analysis to discover the general integral of the Saint-Venant equations when the differential characteristics approach is used, followed by the usage of equations in finite differences.

The answers discovered by mathematical analysis utilizing the idea of small-amplitude waves make up the second category. The solutions derived from the approximate integration of the Saint-Venant equations with their initial substitution with equations in finite differences are included in the third group.

The research of Saint-Venant, S. A. Khristianovich [9], A. A. Arkhangelsky [6], J. Stoker, and others served as the foundation for mathematical modeling of the erratic flow of water in open streams. A.A. Arkhangelsky, Y.D. Guildenblatt, and N.T. Meleschenko developed the characteristics approach suggested by S.A. Khristianovich. The main idea behind this approach is to substitute an equivalent system of four ordinary differential equations for the Saint-Venant system of equations. These equations consist of two direct and inverse characteristic equations as well as two equations that connect the flow elements. We may determine the coordinates t and s of the nodes of the grid of characteristics in the wave plane by solving a system of four equations using infinite differences. The values of the flow parameters, Q(s,t) and z(s,t), are also determined by the computed parameters.

The boundaries of the region of existence of a given solution (a given wave) can be easily obtained using the method of characteristics, especially when tracing the moment and target of a wave transitioning into a continuous one. Additionally, the elements of the mode in boundary sections are calculated using grid methods using the equations in characteristic form. N. A. Kartvelishvili [10] created the variational approach, which is based on using the Bubnov-Galerkin method to calculate the unsteady movement of water. It proved to be practical while utilizing contemporary computers for computations.

N.T. Meleschenko created the small-amplitude wave technique [1, 2].

$$\Delta t = \frac{\Delta x}{|v| + \sqrt{gh}} \quad (9)$$

This limits the use of explicit schemes.
As shown by S.K. Godunov [20], there is another limitation

$$\Delta t \le \frac{c}{g}\sqrt{\frac{R}{i}} = \frac{K}{g\omega\sqrt{i}} \quad (10)$$

where v - flow rate; g - gravitational constant; h - ordinate of the free flow surface; c - Shezi coefficient; R - wetted riverbed perimeter; i - bottom slope; ω - living cross-sectional area; K – flow rate module.

Restriction (11) vanishes for implicit schemes, and the S.K. Godunov (12) condition can be circumvented by obtaining the flow modulus K value from the wave surface's upper layer. Although it has received limited research, experts claim that the finite element method is the best approach for resolving hydraulic problems [21, 22]. N.M. Vernadsky introduced the instant mode approach, which V. A. Arkhangelsky, Y. D. Gildenblat, and I. V. Egiazarov refined (for manual computation) [3, 6]. B. L. Historik created the computer calculation algorithms and programs [23]. One use of the implicit difference scheme for the grid approach is this technique. Here, the instantaneous values of expenses and levels at the boundaries of the computed sections are calculated at each moment of time that corresponds to the end of the predicted time interval (also known as "instantaneous modes"). A scientific school (A.A. Atavin, A.F. Voevodin, M.G. Gladyshev, and S.M. Shugrin) was established in the IG SB RAS in the early 1960s under the direction of Professor O.F. Vasilyev. Their contributions to the mathematical modeling of

unsteady water movement [8, 24, 25]. Mathematical formulations of problems on smoothly changing and discontinuous unsteady motion of water in watercourses and associated systems are provided here, along with equations of motion in a more generic form. Numerical techniques for computing and identifying numerous equation parameters have been developed, as well as for solving one-dimensional problems in stationary and moving grids with and without a gap. Research has been done to improve mathematical modeling's level of automation. The conceptual, analytical, and numerical approaches to issue solving—including the flow of water over a dry riverbed—are given a lot of attention. Both adequate tools and methods for solving two-dimensional problems were established.
A difference boundary value problem with nearly the same structure as the main boundary value problem is solved when differential equations are approximated using implicit difference techniques on each temporal layer. The most promising algorithms for solving difference equations are those that respect this structure [7, 8]. In order to solve such issues, the principal difference equations are used, which are linearized. Then, using one of the sweep method's variations, unknown coefficients are found, and balancing relations are used to establish the system's state parameters.

a comprehensive model of the temporary water flow at a canal site. The Saint-Venant system of differential equations, which takes the form of energy conservation principles, describes the condition of the major canal site, which is characterized by an unstable water flow [27].

$$B\frac{\partial z}{\partial t}+\frac{\partial Q}{\partial x}=q, \tag{11}$$

$$\frac{1}{g\omega}\left(\frac{\partial Q}{\partial t}+2v\frac{\partial Q}{\partial x}\right)+\left(1-\left(\frac{v}{c}\right)^2\right)\frac{\partial z}{\partial x}=$$

$$=\left(i+\frac{1}{B}\left(\frac{\partial \omega}{\partial x}\right)_{h=const}\right)-\left(\frac{v}{c}\right)^2-\frac{Q|Q|}{K^2},$$

where $v=\frac{Q}{\omega}$; $c=\sqrt{\frac{g\omega}{B}}$; $Q=Q(x,t)$ – water flow rate; $z=z(x,t)$ – ordinate of free surface; g – gravitational constant; i - bottom slope; $B=B(z)$ – flow width over a living section surface; $\omega=\omega(z)$– living cross-sectional area; $c=c(z)$ – propagation speed of small waves; $K=K(z)$– flow module.

Partial differential equations of the hyperbolic type in system (13) are the equations of conservation of mass and momentum of the flow, and are a mathematical model of the unsteady movement of water in the open canal.

As functions that determine the flow, the flow rate is selected here *Q(x,t)* and the ordinate of the free surface *z(x,t)*. Independent variables are the longitudinal coordinate *x* and time *t*.

The riverbed is set by the ordinate of the bottom $z_0(x)$ and cross-sectional width *B(x,t)*on distance z (vertical) from the bottom of the riverbed .

Then:

flow depth: $h(z,t) = z(x,t) - z_0(x,t)$; flow cross-sectional area: $\omega(x,t) = \int_0^h B(x,z)dz$; average speed of flow: $\upsilon = \frac{Q}{\omega}$;

small wave propagation speed: $c = \sqrt{g\omega / B}$; bottom slope $i = -dz_0 / dx$.

The characteristic form of equations (13) has the form [7]

$$\frac{\partial Q}{\partial t} + (v \pm c)\frac{\partial Q}{\partial x} - B(v \mp c)\left[\frac{\partial z}{dt} + (v \pm c)\frac{\partial z}{\partial x}\right] = \left(\varphi - \frac{Q|Q|}{2K}\right)g\omega - (v \mp c)q. \tag{12}$$

Here $\varphi = \left[i + \frac{1}{B}\left(\frac{\partial \omega}{\partial x}\right)_{h=const}\right]\left(1 - \frac{v}{c}\right)^2$.

The initial conditions are specified as

$$z(x,0) = z_0(x), \qquad Q(x,0) = Q_0(x), \tag{13}$$

where $Q_0(x)$, $z_0(x)$ – known functions.

Boundary conditions at points $x_1 = 0$ and $x_2 = l$ are written as follows

$$Q(0,t) = u_1(t), \quad Q(l,t) = u_2(t). \tag{14}$$

Water flows at the points of water intake of the canal site, the right side of equation (15) under the conditions of discreteness of water distribution, has the form

$$q(x,t) = -\sum_{i=1}^{5} q_i \delta(x - a_i) l(t - T) \tag{15}$$

In these models, an analytical solution of equations (13), (15) and (16) under the indicated boundary conditions is absent, since the hydraulic parameters of the water flow is a nonlinear function depending on the shape of the cross section of the canal site.

From the expression of the lateral water intakes (17) it is seen that consumers are provided with a discrete water supply in time in the form of a step function. With stepwise functions, to solve the problem of optimal control of water distribution, it is necessary to formulate criteria for the quality of water distribution in the canals of irrigation systems under conditions of discrete water supply to consumers and a system of restrictions.

III. CONCLUSION

The results of an analytical examination of recent and traditional research on mathematical models of irrigation systems' main canals allow for the identification of particular study areas for their application. The analysis of the mathematical models of the primary irrigation system canals takes into consideration the uniform water supply to consumers, which is fundamentally changing the flow rate through the hydraulic structures of the canal. For the country's economy, this is crucial.

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