International Journal of Mathematical Theory and Applications
Vol 1, Issue 2 2025

Updated findings about the bounds and stability of solutions to a certain class of second-order vector differential equations

A. ANJUSREE, CHANDUPRIYA.Y
Department of Mathematics

Article Info
Received: 08-02-2025 Revised:08-03-2025 Accepted:19-03-2025 Published:03-04-2025

Abstract
Using an appropriate Lyapunov function, we prove in this study that for a certain system of second-order differential equations, the zero-solution is stable when P(t, X, Y) = 0 and that all solutions are limited when P(t, X, Y) /= 0. The findings presented in this publication add to and refine previous research. To back up the validity of the findings that have been obtained, examples are provided.

Keywords: Second order, Stability, Boundedness, Lyapunov function.

1 Introduction

Consider the following second order nonlinear vector differential equation given by:

$$\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}) \tag{1.1}$$

or its equivalent system:

$$\dot{X} = Y, \quad \dot{Y} = -AY - H(X) + P(t, X, Y) \tag{1.2}$$

in which $X, Y : \mathrm{R}^n \to \mathrm{R}^n$, $H(X) : \mathrm{R}^n \to \mathrm{R}^n$, $P(t, X, Y) : \mathrm{R}^+ \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^n$, A is an $n \times n$ In the standard format, the dots represent differentiation with respect to t, where t is an element of R+ = [0, infinity], R stands for the real line (−∞, infinity), and Rn is the real n-dimensional Euclidean space with the standard Euclidean norm. The functions H(X) and P (t, X, Y) are taken to be continuous for the explicitly shown parameters, and it is also assumed that H(X) is not necessarily differentiable. For further information, read [1]; we will also assume that solutions to Eq. (1.1) exist and are unique.
There has been a lot of research on the qualitative behavior of solutions to second-order differential equations. In 1976, for example, Tejumola [2] used the second approach.

use Lyapunov's stability and ultimate boundedness tests to determine if a certain matrix differential equation has periodic solutions and, if so, what those solutions are.

$$\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}), \tag{1.3}$$

where A is an $n \times n$ constant matrix and $H(X)$ is a differentiable vector function. Kroopnick ([3], [4]) used the approach of the integral test to prove the stability and boundedness of solutions of a second order scalar linear differential equation

$$x'' + a(t)x = 0, \tag{1.4}$$

where $a(t) > 0$ and $a'(t) \geq 0$. Tunç and Tunç [5] used integral test to investigate the boundedness and stability properties of a second order vector linear differential equation given by

$$\ddot{X} + a(t)X = P(t). \tag{1.5}$$

And recently, in 2017, Tunç and Tunç [6] used Lyapunov's direct method to study the stability and boundedness of a second order nonlinear vector differential equation of the form

$$\ddot{X} + a(t)X = P(t, \dot{X}), \tag{1.6}$$

P (t, $\dot{X}$) also meets the Lipschitz condition in $\dot{X}$, and both a(t) and P (t, $\dot{X}$) are continuous in their arguments. Various more intriguing outcomes pertaining to solutions of second-order and higher-order differential equations have also been attained. For further reading, interested parties can consult the following papers and their cited references: Ademola et al. [7], Adeyanju [8], Afuwape [9], Alaba and Ogundare [10], Cartwright and Littlewood [11], Ezeilo [12], Grigoryan [13], Kroopnick [14], Meng [15], Ogundare and Afuwape [16], Ogundare et al. [17], Ogundare et al. [18], Omeike [19], Omeike et al. [20], Tejumola [21], Tunç [22], Tunç and Tunç [23], [24], [5, 6], Wang and Zhu [25].

2 Notation

Henceforth, δ's, Δ's and K's with or without suffixes will denote positive constants whose magnitudes depend on an $n \times n$ constant matrix A and vector functions $H(X)$, $P(t, X, \dot{X})$. The δ's, Δ's and K's with numerical or alphabetical suffixes shall retain fixed magnitudes, while those without suffixes are not necessarily the same at each occurrence.

We also denote the scalar product $\langle X, Y\rangle$ of any vectors X, Y in R^n, with respective components $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by $\sum_{i=1}^{n} x_i y_i$. In particular, $\langle X, X\rangle = \|X\|^2$.

3 Preliminary results

In this section, we state without proofs some standard results needed in the proofs of our main results.

Lemma 3.1. *Let A be a real symmetric positive definite $n \times n$ matrix. Then, for $X \in \mathrm{R}^n$*

$$\delta_a \|X\|^2 \leq \langle AX, X\rangle \leq \Delta_a \|X\|^2, \tag{3.1}$$

where δ_a and Δ_a are respectively the least and greatest eigenvalues of the matrix A.

Proof. See ([26], [27]) □

Remark 3.2. *It should be note that a similar case where $H(X)$ is not necessarily diflerentiable for a certain third order diflerential equations was considered by [Afuwape [9], Meng [15]].*

4 Stability

In this section, we consider the following equation.

$$\ddot{X} + A\dot{X} + H(X) = 0, \tag{4.1}$$

or its equivalent system:

$$\begin{matrix} \dot{X} \\ Y, \end{matrix} = \dot{Y} = -AY - H(X). \tag{4.2}$$

Theorem 4.1. *Suppose that* $H(0) = 0$ *and that,*

(i) there exists an $n \times n$ *real continuous operator* $B(X, Y)$ *for any vectors* $X, Y \in R^n$ *such that,*

$$H(X) = H(Y) + B(X, Y)(X - Y), \tag{4.3}$$

whose eigenvalues $\lambda_i(B(X, Y)), \quad (i = 1, 2, \ldots, n)$ *satisfy*

$$0 < \delta_b \leq \lambda_i(B(X, Y)) \leq \Delta_b; \tag{4.4}$$

(ii) the constant symmetric matrix A *has positive eigenvalues for any vectors* $X, Y \in R^n$ *and*

$$0 < \delta_a \leq \lambda_i(A) \leq \Delta_a. \tag{4.5}$$

Then, the trivial solution of Eq. (4.1) or system (4.2) is asymptotically stable.

Proof. The main tool in proving our result is the differentiable function $V = V(t, X, Y)$ defined by

$$2V(t, X, Y) = \langle \beta(1-\beta)\delta^2 X, X\rangle + \langle \alpha\delta_a Y, Y\rangle + (1-\beta)^2 \| Y + \delta_a X\|^2, \tag{4.6}$$

where $\alpha > 0, \ 0 < \beta < 1$ are some constants.

Clearly, V(t,0,0) = 0 and

$$2V(t; X, Y) \geq \langle \beta(1-\beta)\delta^2 X, X\rangle + \langle \alpha\delta_a Y, Y\rangle.$$

Thus, there exist a positive constant $K = \min\{\beta(1-\beta)\delta_a^2, \ \alpha\delta_a\}$, such that,

$$2V(t, X, Y) \geq K\{\| X \|^2 + \| Y \|^2\} \tag{4.7}$$

for all $t \geq 0, \ X, \ Y$. From (4.7), $V(t; X, Y) = 0$ if and only if $\|X\|^2 + \|Y\|^2 = 0$ and $V(t; X, Y) > 0$ if and only if $\|X\|^2 + \|Y\|^2 \neq 0$. It then follows that,

$$V(t; X, Y) \longrightarrow +\infty \text{ as } \|X\|^2 + \|Y\|^2 \longrightarrow +\infty.$$

Next, we derive the derivative of V along the trajectories of the system (4.2) and show that it is negative definite for $X \neq 0$ and $Y \neq 0$. By differentiating V, we obtain

$$\begin{aligned} \dot{V}_{(4.2)} &= \langle \beta(1-\beta)\delta_a^2 X, Y\rangle + \langle \alpha\delta_a Y, -AY - H(X)\rangle + (1-\beta)^2\langle Y + \delta_a X, -AY - H(X) + \delta_a Y\rangle \\ &\leq -\alpha\delta_a^2\langle Y, Y\rangle - (1-\beta)^2\delta_a\langle X, H(X)\rangle + (1-\beta)\delta_a^2\langle X, Y\rangle - (1-\beta)^2\delta_a\langle X, AY\rangle \\ &\quad - \alpha\delta_a\langle Y, H(X)\rangle - (1-\beta)^2\langle Y, H(X)\rangle \\ &\leq -\alpha\delta_a^2\langle Y, Y\rangle - (1-\beta)^2\delta_a\langle X, H(X)\rangle - \{\alpha\delta_a + (1-\beta)^2\}\langle Y, H(X)\rangle \\ &\quad - \langle \delta_a(1-\beta)\{A(1-\beta) - \delta_a I\}X, Y\rangle. \end{aligned}$$

where

$$U_1 = \frac{1}{2}\alpha\delta_a^2\langle Y, Y\rangle + \frac{1}{2}(1-\beta)^2\ \delta_a\langle X, H(X)\rangle,$$

$$U_2 = \frac{1}{4}\alpha\delta_a\langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\ \delta_a\langle X, H(X)\rangle + \langle\delta_a(1-\beta)\{A(1-\beta)-\delta_a I\}X, Y\rangle$$

and

$$U_3 = \frac{1}{4}\alpha\delta_a^2\langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\ \delta_a\langle X, H(X)\rangle + \{\alpha\delta_a + (1-\beta)^2\}\langle Y, H(X)\rangle.$$

In what follows, we will show that $U_2 \geq 0$ and $U_3 \geq 0$. Let K_1 and K_2 be any two strictly positive constants which are carefully chosen, we have

$$\begin{aligned}&\langle\delta_a(1-\beta)\ A(1-\beta)-\delta_a I\ X, Y\rangle\\ &= \|K_1^{-1}\delta_a(1-\beta)(A(1-\beta)-\delta_a I)X + \frac{1}{2}K_1 Y\ \|^2 - K_1^{-2}\delta_a^2(1-\beta)^2\ A(1-\beta)-\delta_a I\ ^2\|X\|^2 - \frac{1}{4}K_1^2\|Y\|^2\\ &\geq -K_1^{-2}\ \delta_a^2(1-\beta)^2\ A(1-\beta)-\delta_a I\ X\ - \frac{1}{4}K_1^2\|Y\|^2\end{aligned}$$

and

$$\begin{aligned}&\langle(\alpha\delta_a + (1-\beta)^2)H(X), Y\rangle\\ &= \|K_2^{-1}(\alpha\delta_a + (1-\beta)^2)H(X) + \frac{1}{2}K_2 Y\|^2 - K_2^{-2}\ (\alpha\delta_a + (1-\beta)^2)\ \|H(X)\|^2 - \frac{1}{4}K_2^2\|Y\|^2\\ &\geq -K_2^{-2}\ (\alpha\delta_a + (1-\beta)^2)\ \|H(X)\|^2 - \frac{1}{4}K_2^2\|Y\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}U_2 &\geq -\frac{1}{4}K_1^2\|Y\|^2 - K_1^{-2}\delta_a^2(1-\beta)^2\ A(1-\beta)-\delta_a I\ ^2\|X\|^2 + \frac{1}{4}\alpha\delta_a^2\langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\delta_a\langle X, H(X)\rangle\\ &\geq \frac{1}{4}\ \alpha\delta_a^2 - K_1^2\ \langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\delta_a\ \delta_b - 4K_1^{-2}\delta_a\Delta_a^2\beta^2\ \langle X, X\rangle,\end{aligned}$$

where we have applied Eq. (4.3) with $Y = 0$, $H(Y) = 0$ and Lemma 3.1. Therefore, for all $X, Y \in \mathrm{R}^n$,
$U_2 \geq 0$,
if $K_1^2 \leq \alpha\delta_a^2$ with

$$\Delta_b^2 \leq \frac{1}{4\delta_a\beta^2} \leq \frac{\alpha\delta_a\delta_b}{4\beta^2}.$$

Also,

$$U_3 \geq \frac{1}{4}\alpha\delta_a\langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\ \delta_a\langle X, H(X)\rangle - K_2^{-2}\ (\alpha\delta_a + (1-\beta)^2)\ \|H(X)\|^2 - \frac{1}{4}K_2^2\|Y\|^2.$$

On applying Eq. (4.3) with $Y = 0$, $H(0) = 0$ and Lemma 3.1, we have

$$U_3 \geq \frac{1}{4}(\alpha\delta_a^2 - K_2^2)\langle Y, Y\rangle + \frac{1}{4}\ \delta_a\delta_b(1-\beta)^2 - \frac{\alpha\delta_a + (1-\beta)^2\Delta_b^2}{K_2^2}\ \langle X, X\rangle.$$

Thus, for all $X, Y \in R^n$,

$$U_3 \geq 0,$$

if $K^2 \leq \alpha\delta^2$ with

$$\Delta_b^2 \leq \frac{K^2\delta_a\delta_b(1-\beta)^2}{4\ \alpha\delta_a + (1-\beta)^2} \leq \frac{\alpha\delta^3\delta_b(1-\beta)^2}{4\ \alpha\delta_a + (1-\beta)^2}.$$

Lastly, we are left with the estimate for U_1. From (4.8) and (4.3), we have

$$U_1 = \frac{1}{2}\alpha\delta_a\langle Y, Y\rangle + \frac{1}{2}(1-\beta)\delta_a\langle X, H(X)\rangle$$
$$\frac{1}{2}\alpha\delta_a\langle Y, Y\rangle + \frac{1}{2}(1-\beta)\delta_a\langle X, BX\rangle$$
$$\frac{1}{2}\alpha\delta_a\langle Y, Y\rangle + \frac{1}{2}(1-\beta)\delta_a\delta_b\langle X, X\rangle$$
$$\geq \delta_2\{\|X\|^2 + \|Y\|^2\},$$

where $\delta_2 = \frac{1}{2}\min\{\alpha\delta_a^2; (1-\beta)^2\delta_a\delta_b\}$.

Therefore,

$$\dot{V}_{(4.2)} \leq -\delta_2\{\|X\|^2 + \|Y\|^2\} \leq 0. \tag{4.9}$$

So far, we have been able to show that the trivial solution is stable. Now, consider the set defined by $W = \{(X, Y) : \dot{V}(X, Y) = 0\}$.

By applying LaSalle's invariance principle (see [5]), we observe that $(X, Y) \in W$ implies that

$$X = Y = 0. \tag{4.10}$$

This, i.e (4.10), shows that the largest invariant set contained in W is $(0, 0) \in W$. Therefore, we conclude that the trivial solution of system (4.2) or Eq. (4.1) is asymptotically stable and this completes the proof of Theorem 4.1. □

Example 4.2. *As a special case of equation (1.1) with $n = 2$, we provide the following example. In (1.1), let*

$$X = \begin{matrix} x_1 \\ x_2 \end{matrix}, \quad A = \begin{matrix} 4 & 1 \\ 1 & 4 \end{matrix} \quad \text{and} \quad H(X) = \begin{matrix} b_1 & x_1 \\ b_2 & \frac{b_2}{b_1}BX, \end{matrix} \quad x_2$$

where b_1, b_2 are constants satisfying $0 < b_2 < b_1$. Then, by some simple calculations, $0 < \delta_a = 3 \leq \lambda_i(A) \leq \Delta_a = 5$, and $0 < \delta_b = b_1 - b_2 \leq \lambda_i(B) \leq \Delta_b = b_1 + b_2$.
Thus, all the conditions of the theorem are satisfied.

Remark 4.3. *Meng [15], considered the Ultimate boundedness results of a certain third order nonlinear linear diflerential equation in which the nonlinear term is not necessary diflerentiable and gave a similar example to our Example 4.2 above.*

5 Boundedness

We now state and prove the result on the boundedness of solutions for the case $P(t, X, Y) \neq 0$.

Theorem 5.1. *Suppose in addition to the conditions (i) and (ii) of Theorem 4.1, the following holds:*

(iii) for all t, X and Y, there exist a finite constant $\delta_1 > 0$ and a continuous function $\theta = \theta(t)$ such that the vector $P(t, X, Y)$ satisfies

$$\|P(t, X, Y)\| \leq \theta(t), \tag{5.1}$$

where $\int_0^t \theta(s)ds \le \delta_1 < \infty$ *for all* $t \ge 0$. *Then there exists a constant* $D > 0$ *such that every solution* $(X(t), Y(t))$ *of* (1.2) *determined by*

$$X(0) = X_0, \quad Y(0) = Y_0$$

satisfies

$$\|X(t)\| \le D, \quad \|Y(t)\| \le D \tag{5.2}$$

for all $t \ge 0$, $X, Y \in \mathrm{R}^n$.

Proof. To establish this result, we make use of the Lyapunov function defined in (4.6). We note that under the assumptions of Theorem 5.1, the estimate (4.7) for $V(t, X, Y)$ is still the same. But now that $P(t, X, Y) \not\equiv 0$, we obtain the following as the derivative of V:

$$\begin{aligned}
\dot{V}_{(1.2)} &\le -\delta_2\{\|X\|^2 + \|Y\|^2\} + \langle\{\alpha\delta_a + (1-\beta)^2\}Y + (1-\beta)^2\delta_a X, P\rangle \\
&\le \{\alpha\delta_a + (1-\beta)^2\}\|Y\| + (1-\beta)^2\delta_a\|X\| \|P(t, X, Y)\| \\
&\le \delta_3\{\|X\| + \|Y\|\}\|P(t, X, Y)\| \\
&\le \delta_3\theta(t)\{\|X\| + \|Y\|\}.
\end{aligned}$$

where $\delta_3 = \max\{\{\alpha\delta_a + (1-\beta)^2\}; (1-\beta)^2\delta_a\}$. On applying the inequalities $\|X\| \le 1 + \|X\|^2$ and $\|Y\| \le 1 + \|Y\|^2$, we have

$$\begin{aligned}
\dot{V}_{(1.2)} &\le \delta_3\theta(t)\{2 + \|X\|^2 + \|Y\|^2\} \\
&= 2\delta_3\theta(t) + \delta_3\theta(t)\{\|X\|^2 + \|Y\|^2\}.
\end{aligned}$$

By using the inequality (4.7), we have

$$\dot{V}(t) \le 2\delta_3\theta(t) + 2\delta_3\theta(t)K^{-1}V_{(1.2)}(t, X, Y).$$

Integrating the last inequality from 0 to t, $(t > 0)$, we obtain

$$\begin{aligned}
\frac{V(t) - V(0)}{2\delta_3} &\le \int_0^t \theta(s)ds + K^{-1}\int_0^t \theta(s)V(s)ds \\
&\le 2\delta_1\delta_3 + \delta_3 K^{-1}\int_0^t \vartheta(s)V(s)ds.
\end{aligned}$$

If we let $2\delta_1\delta_3 + V(0) = \delta_4$ and $2\delta_3K^{-1} = \delta_5$, we obtain

$$V(t) \le \delta_4 + \delta_5\int_0^t \vartheta(s)V(s)ds.$$

Applying Grownwall Bellman's inequality (see Rao [1]) in the above inequality, we get

$$V(t) \le \delta_4 \exp\left[\delta_5\int_0^t \theta(s)ds\right] \le \delta_4\exp(\delta_6) \tag{5.3}$$

where $\delta_6 = \delta_1\delta_5$.
Thus, (5.3) implies that all the solutions of the Eq. (1.1) or it's equivalent system (1.2) are bounded. □

Example 5.2. *Suppose, in addition to Example 1, we have*

$$P(t; X, Y) = \begin{pmatrix} \frac{1}{1+e^t+x_1^2+y_1^2} \\ \frac{1}{1+e^t+x_2^2+y_2^2} \end{pmatrix}.$$

It is very clear that

$$\|P(t; X, Y)\|_2 \leq = \theta(t) \text{ and } \max \theta(t) = 2 < \infty.$$

$$e^t \int_0^\infty \frac{1}{e^s} ds = 2.$$

$$\int_0^\infty \vartheta(s) ds = 2$$

Also, all the conditions of the theorem are satisfied.

Remark 5.3. *We have thus far, established stability of the trivial solution and boundedness of all solutions of the systems* (4.2) *and* (1.2) *respectively without imposing diflerentiability condition on the vector function H*(*X*) *as in papers ([27], [2]).*

References

[1] [1] Rao, M.R. Ordinary Differential Equations. Affiliated East West Private Limited London, (1980).
In the Proceedings of the Accademia Nazionale Dei Lincei Serie VII, volume 60, issue 2, pages 100–107 (1976), H.O. Tejumola established boundedness criteria for the solutions of a number of second-order differential equations.
Krupnick, A.J., "Bounded solutions to x″ + q(t)b(x) = f (t)" [3]. Pages 829–836 of the International Journal of Mathematical Education in Science and Technology, volume 41, issue 6, published in 2010.
Four additional demonstrations of the boundedness of solutions to the equation x″ + a(t)x = 0 (Kroopnick, A.J. 2014). The state of Missouri.
Published in 2013, the article is published in the Journal of Mathematical Sciences, volume 25, issue 1, pages 103–105.
[5] A remark on specific qualitative features of a second-order linear differential system by Tunç, C. and Tunç, O. International Journal of Mathematical Sciences, Volume 9, Issue 2, Pages 953–956, (2015).
The stability and boundedness of solutions to second-order non-linear differential systems are discussed in a note by Tunç & Tunç [6]. “Journal of the Association of Arab Universities for Basic and Applied Sciences” 24, 169–175, (2017).
Adesina, O.A., Ogundiran, M.O., and Ademola, A.T. Periodicity, stability, and solutions to certain second-order delay differential equations are discussed in [7]. 10 pages, 2016 edition, published in the International Journal of Differential Equations, Article ID 2843709.
For a certain class of second-order non-linear vector differential equations, there exists a limiting regime in the demidovic meaning [8] Adeyanju, A.A. In the 2018 edition of Diflerential Equations and Control Processes, the paper is numbered 63–79.
According to Afuwape (A. U.), some third-order nonlinear differential equations have ultimate boundedness findings [9]. Vol. 97, Issues 140–150, Journal of Mathematical Analysis and Applications (1983).
Resolving specific second-order non-autonomous nonlinear ordinary differential equations: stability and boundedness features [10] Alaba and Ogundare. Journal of Mathematics in Kragujevac 39, 255-266 (2015).
Regarding second-order nonlinear differential equations, Cartwright and Littlewood [11]. Mathematical Annals 48, 472-494 (1947).

Upon the convergence of solutions of certain systems of second-order differential equations [12] Ezeilo, J.O.C. Applied Mathematics, Volume IV, Section II, Pages 239–252, 1966.
The article by Grigoryan, G.A. discusses stability requirements and boundedness for second-order linear ordinary differential equations. Volume 57, Issue 12, pages 2–15, Russian Mathematics (2013).
Int. Math Forum 9(10), 475-481 (2014), Kropnick, A.J. discussed the integration of L2-solutions of non-oscillatory solutions to x″ + a(t)x′ + k2x = 0.
In a specific system of third-order nonlinear differential equations, Meng F. W. proved the ultimate

bounds. Page 496–509 of the 1993 Journal of Mathematical Analysis and Applications volume 177. Citation: [16] B. S. Ogundare and A. U. Afuwape. Constraints and stability characteristics of generalized Lienard equation solutions. Journal of Mathematics from Kochi, Volume 9, Issue 4, Pages 97–108, 2014, (2014).

According to Ogundare, B. S., NGCIBI, and V. Murali (17), solutions to certain second-order differential equations exhibit boundedness and stability characteristics. Volume 3, Issues 38–50, Advances in Differential Equations and Control Processes (2015).

"On the qualitative behavior of solutions to certain second order nonlinear differential equation with delay" [18] is the work of B.S. Odundare, A.T. Ademola, M.O. Ogundiran, and O.A. Adesina. Annual Report of the University of Ferrara 63, 333–351, 2017.

The stability and boundedness of third-order nonlinear vector differential equations are discussed in Omeike's work [19]. Paper published in 2014 in the Archivum Mathematicum, volume 50, pages 101–106.

The stability and boundedness of solutions of certain vector delay differential equations were discussed by Omeike, Adeyanju, and Adams (2021). Section 37(2), pages 77–87, published in 2018 by the Nigerian Mathematical Society.

The boundedness theorems for certain systems of two differential equations are given by Tejumola, H.O. [21] in Atti. Published in 1971 in the Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., volume 51, issue 6, pages 472-476.

For third-order nonlinear vector differential equations, Tunç, C. [22] discusses the stability and boundedness of solutions. The referenced work is Nonlinear Analysis 70(6), pages 2232-2236, published in 2009.

In their 2007 article "On the asymptotic behaviour of solutions of certain second order differential equations," Tunç and Tunç discussed the topic in the Journal of the Franklin Institute, volume 344, pages 391-398.

In their 2011 article published in the Sarajevo Journal of Mathematics, Tunç and Tunç discussed the boundedness of solutions to second-order non-autonomous differential equations [24]. [25] Wang and Zhu: Periodic solutions of a Duffing equation under periodic and anti-periodic eigenvalue conditions: existence, uniqueness and stability (2015) in the Taiwanese Journal of Mathematics, volume 19, issue 5, pages 1457–1468.

In their publication titled "Further results for a system of third-order ordinary differential equations," Ezeilo and Tejumola published their work in [26]. The article was published in the Annals of the Lincei Academy of Science, volume 58, pages 143–151 (1975).

Boundedness of solutions of certain systems of second-order ordinary differential equations [27] Omeike, M.O., Oyetunde, O.O., and Olutimo, A.L. The article is published in the mathematical journal Acta Univ. Palacki Olomuc., volume 53, pages 107–115, 2014.