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Application of the Boubakar collocation method to the numerical solution of integral and integralo-differential equations with linear parameters

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Abstract

In order to solve first-order linear integral and integro-differential equations with starting conditions, this work presents a novel collocation approach that is based on Boubakar approximation polynomial. After transforming the integro-differential equations into integral equations, the usual collocation technique is used to turn them into a system of linear equations. After that, the matrix inversion approach is used to solve the linear equation. The approach is shown to be efficient in addressing the issues under investigation using numerical solutions in three cases.

Keywords and phrases: integral equations, Fredholm-Volterra integro-diflerential equations, collocation method, Boubakar polynomial

1 Introduction

More and more, researchers are concentrating on finding better ways to solve integral and integro-diflerential equations, which are employed in many fields including ocean circulation, electromagnetic theory, and heat and mass diflusion models [1]. The following numerical methods have been developed for solving integral and integro-diflerential equations: hybrid linear multistep method [2], Daftardar-Gejji and Jafari method [3], collocation method [1, 4, 5, 6], block by block method [7], shifted Chebyshev polynomial of the third kind [8], Chebyshev-Gelarkin method [9], diflerential transform [10], Adomian decomposition method [11], Chebyshev collocation method [12], Bernoulli matrix method [13], Berstein operational matrix [14], pseudospectral method using Legendre polynomial [15], to name a few. Creating a novel collocation approach to solving linear Fredholm-Volterra integral equations is the focus of this work.

$$y(x) = f(x) + \lambda_1 \int_0^x k(x,t)\, y(t)\, dt + \lambda_2 \int_0^1 w(x,t)\, y(t)\, dt \tag{1.1}$$

and integro-diflerential equations

$$y'(x) = f(x) + \lambda_1 \int_0^x k(x,t)\, y(t)\, dt + \lambda_2 \int_0^1 w(x,t)\, y(t)\, dt, \quad y(0) = y_0 \tag{1.2}$$

$a \le x \le b$, $y(x)$ is the unknown function, $f(x)$ and $k(x,t)$, $w(x,t)$ are given continuous real valued functions, using the approximate solution

$$y_n(x) = \mathbf{B}(x)\,\mathbf{A} \tag{1.3}$$

where $\mathbf{B}(x)$ is the Boubakar interpolating polynomial with the recursive formula

$$B_{n+1}(x) = xB_n(x) - B_{n-1}(x),\ n \geq 2 \tag{1.4}$$

where $\mathbf{B}(x) = \begin{bmatrix} B_0(x) & B_1(x) & \cdots & B_N(x) \end{bmatrix}_{1\times(N+1)}$, $B_0(x) = 1$, $B_1(x) = x$, $B_2(x) = x^2 + 2$ and $\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T_{(N+1)\times 1}$ are constants to be determined

2 Methodology

In this section, we discuss the steps in developing the new collocation method. We first convert the integro-diflerential equation to integral equation, hence, (1.2) can be written in the form

$$y(x) = y_0 + \int_0^x f(t)\,dt + \lambda_1 \int_0^x \int_0^x k(x,t)\,y(t)\,dt\,dx + \lambda_2 \int_0^x \int_0^1 w(x,t)\,y(t)\,dt\,dx \tag{2.1}$$

we substitute (1.3) into (2.1) to give

$$\mathbf{w}(x)\,\mathbf{A} = g(x) \tag{2.2}$$

where

$$\mathbf{w}(x) = \mathbf{B}(x) - \lambda_1 \int_0^x \int_0^x k(x,t)\,\mathbf{B}(t)\,dt\,dx - \lambda_2 \int_0^x \int_0^1 w(x,t)\,\mathbf{B}(t)\,dt\,dx \quad {}_{1\times(N+1)}$$

$$g(x) = y_0 + \int_0^x f(t)\,dt$$

We then collocate (2.2) at

$$x_i = a + \frac{(b-a)\,i}{N} \tag{2.3}$$

to give

$$\mathbf{w}(x_i)\,\mathbf{A} = \mathbf{g}(x_i) \tag{2.4}$$

with the dimension $\mathbf{w}(x_i) = (N+1) \times (N+1)$, $\mathbf{g}(x_i) = (N+1) \times 1$. We then solve for the constants in (2.4) using matrix inversion method and substitute the result into the approximate solution (1.3) to get the numerical solution

3 Convergence Analysis

In this section, we establish the convergence of the new method. We substitute the approximate solution into (1.2)

$$y_N(x) = y_0 + \int_0^x f(t)\,dt + \lambda_1 \int_0^x \int_0^x k(x,t)\,y_N(t)\,dt\,dx + \lambda_2 \int_0^x \int_0^1 w(x,t)\,y_N(t)\,dt\,dx \quad (3.1)$$

Substract (1.2) from (3.1) and using

$$E_N(x) = y_N(x) - y(x) \quad (3.2)$$

then

$$|E_N(x)| \le |\lambda_1| \int_0^x \int_0^x k(x,t)\,E_N(t)\,dt\,dx + |\lambda_2| \int_0^x \int_0^1 w(x,t)\,E_N(t)\,dt\,dx$$

hence

$$\frac{\|E_N(x_i)\|_\infty}{\|E_N(t)\|_\infty} \le |\lambda_1| \int_0^{x_i} \int_0^{x_i} k(x,t)\,dt\,dx_i + |\lambda_2| \int_0^{x_i} \int_0^1 w(x_i,t)\,dt\,dx_i$$

hence, the method converges

4 Numerical Examples

In this section, we solve three examples to test the efficiency of the new method. All the results are presented in tables, all computations are done with the aid of program written in MATLAB (2015a) and run on a PC. abs-e_N = $|y_N(x) - y(x)|$ use in the tables refers to the absolute error for N, $y_N(x)$ is the numerical solution and $y(x)$ is the exact solution

Example 4.1. /5/ Consider the linear Volterra Fredholm inte9ro-difierential equation

$$y'(x) = -f(x) - \int_0^x x^2 t y(t)\,dt + \int_0^1 x(x-t)\,y(t)\,dt\,, \quad y(0) = 1, 0 \le x \le 1 \quad (4.1)$$

$$f(x) = \sin x - x^2\cos x - x^3\sin x + x^2 - x^2\sin(1) + x\cos(1) + x\sin(1) - x$$

/nte9ratin9 (4.1) from 0 to x 9ives

$$y(x) = y(0) - \int_0^x f(t)\,dt - \int_0^x \int_0^x x^2 t y(t)\,dt\,dx - \int_0^x \int_0^1 x(x-t)\,y(t)\,dt\,dt \quad (4.2)$$

We solve this problem usin9 N—5 and 10 but we use N —5 for illustration. Łet the approzimate solution (1.3) reduces to the form $\mathbf{B}(x) = \begin{matrix} B_0(x) & B_1(x) & B_2(x) & B_3(x) & B_4(x) & B_5(x) \end{matrix}$

and $\mathbf{A} = \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix}^T$. 5ubstitutin9 into (4.2), (2.2) reduces to

$$\mathbf{w}(x) = \mathbf{B}(x) + \int_0^x \int_0^x x^2 t\,\mathbf{B}(t)\,dt\,dx + \int_0^x \int_0^1 x(x-t)\,\mathbf{B}(t)\,dt\,dt$$

and

$$g(x) = 1 - \int_0^x f(t)\,dt$$

We then collocate at

$$x_i = \begin{matrix} 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{matrix}$$

then, the parameters in (2.4) 9ive

$$\mathbf{w}(x_i) = \begin{bmatrix} 1 & 0 & \frac{-2}{3} & 0 & \frac{8}{15} & 0 \\ \frac{186131}{187500} & \frac{54751}{281250} & \frac{-14221861}{23625000} & \frac{-12133487}{337500000} & \frac{593103319}{1687500000} & \frac{251288807}{585937500} \\ \frac{46048}{46875} & \frac{54032}{140625} & \frac{-2419331}{5906250} & \frac{-6125971}{10546875} & \frac{-39323803}{421875000} & \frac{728904659}{1757812500} \\ \frac{61861}{62500} & \frac{18081}{31250} & \frac{-244609}{2625000} & \frac{-6628041}{12500000} & \frac{-93557237}{187500000} & \frac{-5622906}{48828125} \\ \frac{48911}{46875} & \frac{111548}{140625} & \frac{1018852}{2953125} & \frac{-812251}{10546875} & \frac{-38755979}{105468750} & \frac{-145726559}{292968750} \\ \frac{71}{60} & \frac{19}{18} & \frac{6911}{7560} & \frac{793}{861} & \frac{21667}{21600} & \frac{931}{900} \end{bmatrix}$$

and

$$\mathbf{g}(x) = \begin{bmatrix} 1.0 & 0.974707 & 0.909465 & 0.824486 & 0.747276 & 0.712805 \end{bmatrix}^T$$

5olvin9 for the unknown constants **A** in (2.4) usin9 matriz inversion 9ives

$$\mathbf{A} = \begin{bmatrix} 0.812324 & -0.0035197 & -0.274497 & -0.00236426 & 0.00877066 & -0.000411614 \end{bmatrix}^T$$

We then substitute **A** into the approzimate solution to 9ive the numerical solution

$$y_5(x) = -3.9618 \times 10^{-3} x^5 + 4.6046 \times 10^{-2} x^4 - 2.2857 \times 10^{-3} x^3 - 0.49945 x^2 - 4.8104 \times 10^{-5} x + 1$$

Followin9 the steps discussed earlier, we solve for N—10 to obtain

$$y_{10}(x) = -2.4083 \times 10^{-7} x^{10} - 1.1041 \times 10^{-7} x^9 + 2.4971 \times 10^{-5} x^8 - 1.5683 \times 10^{-7} x^7 - 1.3888 \times 10^{-3} x^6 - 3.6103 \times 10^{-8} x^5 + 4.1667 \times 10^{-2} x^4 - 1.3907 \times 10^{-9} x^3 - 0.5 x^2 - 3.9712 \times 10^{-12} x + 1$$

Evaluation of the numerical solution at some selected points is 9iven is Table 1 with the comparison with the ezact solution. Table 1 shows that as N is increasin9, the error is decreasin9, hence the accuracy is increasin9. This shows that the method conver9es

Table 1: Comparison of exact and numerical solutions for (4.1)

	Exact	Present method			
x_i	$\cos(x_i)$	$N = 5$	abs-e$_5$	$N = 10$	abs-e$_{10}$
0.2	0.980066577841	0.980066576138695	1.7025e-09	0.980066577840337	9.0503e-13
0.4	0.921060994002	0.921060989870354	4.1325e-09	0.921060994002161	7.2389e-13
0.6	0.825335614909	0.825335611967181	2.9425e-09	0.825335614909179	4.9911e-13
0.8	0.696706709347	0.696706714328259	4.9811e-09	0.696706709346956	2.0923e-13
1.0	0.540302305868	0.540302337522625	3.1654e-08	0.540302305868027	1.1295e-13

Example 4.2. /5/, Consider linear Volterra Fredholm inte9ro-difierential equation

$$y'(x) = 2e^x - 2 + \int_0^x y(t)\,dt + \int_0^1 y(t)\,dt, \quad y(0) = 0, 0 \le x \le 1 \tag{4.3}$$

Writin9 in the inte9ral form 9ives

$$y(x) = \int_0^x \left(2e^t - 2\right)\,dt + \int_0^x \int_0^x y(t)\,dt\,dx + \int_0^x \int_0^1 y(t)\,dt\,dx \tag{4.4}$$

We the solve (4.4) for N—5 and 10, the numerical solutions are:

$$y_5(x) = 0.076\,43x^5 + 0.123\,31x^4 + 0.524\,x^3 + 0.994\,01x^2 + 1.\,000\,5x + 1.\,387\,8 \times 10^{-17}$$

$$\begin{aligned} y_{10}(x) = {} & 4.\,794\,3 \times 10^{-6}\,x^{10} + 1.\,945\,3 \times 10^{-5}\,x^9 + 2.\,060\,9 \times 10^{-4}\,x^8 \\ & + 1.\,382 \times 10^{-3}\,x^7 + 8.\,337\,3 \times 10^{-3}\,x^6 + 4.\,166\,5 \times 10^{-2}\,x^5 \\ & + 0.166\,67x^4 + 0.5x^3 + x^2 + x + 2.\,273\,7 \times 10^{-13} \end{aligned}$$

Comparison of the ezact solution and the numerical solution is 9iven in Table 2. /t is equally shown that the method is efcient and conver9ent

Table 2: Comparison of exact and numerical solutions for (4.3)

	exact	Present method			
x_i	$x_i e^{x_i}$	N=5	abs-e$_5$	N=10	abs-e$_{10}$
0.2	0.24428055163203	0.2442824737318	1.9221e-06	0.244280551632	7.8606e-13
0.4	0.59672987905650	0.5967338938959	4.0148e-06	0.596729879057	1.3940e-12
0.6	1.09327128023431	1.0932775362937	6.2561e-06	1.093271280236	2.0199e-12
0.8	1.78043274279397	1.7804414752188	8.7324e-06	1.780432742796	2.5655e-12
1.0	2.71828182845905	2.7182934980576	1.1670e-05	2.718281828462	2.9569e-12

Example 4.3. /17/ Consider the inte9ral equation

$$y(x) = e^x - 1 - x + \int_0^x y(t)\,dt + \int_0^1 xy(t)\,dt, \quad 0 \le x \le 1 \tag{4.5}$$

We solve this problem usin9 N—5 and 10, the numerical solutions 9ive

$$\begin{aligned} y_5(x) = {} & 7.\,673\,1 \times 10^{-2}\,x^5 + 0.122\,56x^4 + 0.524\,68x^3 + 0.993\,76x^2 + 1.\,000\,6x \\ & + 6.\,938\,9 \times 10^{-18} \end{aligned}$$

$$\begin{aligned} y_{10}(x) = {} & 4.\,798\,7 \times 10^{-6}\,x^{10} + 1.\,943\,1 \times 10^{-5}\,x^9 + 2.\,061\,4 \times 10^{-4}\,x^8 \\ & + 1.\,382\,0 \times 10^{-3}\,x^7 + 8.\,337\,4 \times 10^{-3}\,x^6 + 4.\,166\,5 \times 10^{-2}\,x^5 \\ & + 0.166\,67x^4 + 0.5x^3 + 1.0x^2 + x - 5.\,684\,3 \times 10^{-14} \end{aligned}$$

Evaluation of numerical solution at some selected points is 9iven in Table 3. /t is shown clearly that the method is conver9ent and efcient in handlin9 the problem

Table 3: Comparison of the exact and numerical solution for (4.5)

	exact	present method			
x_i	$x_i e^{x_i}$	N=5	abs-e_5	N=10	abs-e_{10}
0.2	0.24428055163203	0.2442872626330	6.7110e-06	0.24428055163263	5.9857e-13
0.4	0.59672987905650	0.5967418857557	1.2007e-05	0.59672987905751	1.0064e-12
0.6	1.09327128023431	1.0932909320262	1.9652e-05	1.09327128023544	1.1332e-12
0.8	1.78043274279397	1.7804605583613	2.7816e-05	1.78043274279503	1.0507e-12
1.0	2.71828182845905	2.7183224860943	4.0658e-05	2.71828182845989	8.3978e-13

5 Conclusion

In order to solve integro-differential equations and first-order Fredholm Volterra integrals, we have created a novel collocation approach. When compared to other approaches, the one utilized in this research is simpler to develop and has a lower computational overhead. The method's efficiency in addressing the issues under examination is confirmed by the numerical solutions presented in the tables. Furthermore, the technique has strong stability, as seen by the narrow gap between the maximum and minimum errors.

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