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To Nonlinear Impulsive Differential Equations and Vector Lyapunov Functions with Uniform Eventual Stability

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Abstract

We consider a family of piecewise continuous functions that generalise the vector Lyapunov functions and study the uniform eventual stability of nonlinear impulsive differential equations with fixed moments of impulse. Along with the findings of the comparison, we also provide the necessary requirements for the final stability to be uniform. The findings broaden the scope of the existing research on uniform eventual stability analysis beyond the narrower scalar example.

Keywords: Uniform Eventual Stability, Impulsive Differential Equation, Lyapunov Method.

1 Introduction

The occurrence of sudden changes in condition is a hallmark of many evolutionary processes. In the grand scheme of things, the length of the disruptions experienced by these processes is negligible. Consequently, it makes sense to characterise these disturbances as impulses, which are instantaneous events. Optimal control models in economic theory, pharmacokinetic models, frequency-modulated systems, biological processes involving threshold dynamics, and burst-ing rhythm models are just a few examples of the many applications where this impulsive character is on display [1]. The study of dynamic systems affected by abrupt, transitory changes at discrete moments in time has centred on impulsive differential equations. Mechanical shocks, biological cycles, and economic swings are all examples of sudden changes that make stability analysis more complicated than it already is. According to [2–13], Lyapunov stability theory is a crucial tool for assessing system stability within this context. Nonlinear impulsive differential equations exhibit complicated stability behaviours that are difficult for conventional methods to capture. That is why impulsive differential equations provide a solid and practical basis for simulating complicated real-world events that

standard differential equations do not adequately characterise. In systems where impulses substantially impact the behaviour of the system over time, this framework offers an invaluable viewpoint on the dynamics and stability of the system.

The emphasis is on ultimately stable sets that are asymptotically invariant rather than an equilibrium or invariant point in many perturbation and adaptive control issues, as mentioned in [14]. From this vantage point, we may see Lyapunov stability as an example of eventual stabilities more generally. The qualitative characteristics of impulsive differential equations have attracted a lot of attention from scholars for a long time, as shown by papers in [1, 15-20].
The ultimate stability of impulsive differential systems under constrained perturbations is addressed in [26] with respect to the stability of perturbed differential equations. Exactly, using vanishing perturbations and piecewise continuous auxiliary functions that generalise traditional Lyapunov functions, [14] established enough conditions to maintain uniform eventual stability in impulsive differential systems with non-fixed impulse moments. The work in [21] gave findings on the uniform eventual stability for impulsive differential systems with bounded perturbations and non-fixed impulse moments, which is an additional advancement. In impulsive systems, where adaptive control and perturbation factors need a more general analytical approach than traditional Lyapunov stability, these findings highlight the important significance of eventual stability.

Using a class of piecewise continuous auxiliary functions to generalise vector Lyapunov functions, we investigate the uniform eventual stability of impulsive differential systems in this study. Using these generalised Lyapunov functions in conjunction with comparison findings, we prove that solutions in these systems will eventually be uniformly stable. Additionally, a practical example is given to show how the suggested stability criteria work and how effective they are.
In order to account for impul-sive effects and non-fixed moments in dynamic systems, our method presents a more flexible stability analysis framework.

2 Preliminaries, Notations and Definitions

Let R^n denote the n-dimensional Euclidean space with norm $\|.\|$, and let Ω represent a domain within R^n that contains the origin. Define the sets $\mathsf{R}_+ = [0, \infty)$ and $\mathsf{R} = (-\infty, \infty)$, with an initial time $t_0 \in \mathsf{R}_+$ and $t > 0$.

Consider a subset $J \subset \mathsf{R}_+$. We define the class of functions $PC[J, \Omega]$ as those functions $\alpha : J \to \Omega$ such that $\alpha(t)$ is piecewise continuous with potential points of discontinuity $t_k \in J$, at which $\alpha(t)$ is well-defined.

Now, consider the following impulsive differential system:

$$\begin{aligned} \mu' &= \Xi(t, \mu), \quad t \geq t_0, \ t \neq t_k, \ k = 1, 2, \ldots, \\ \Delta\mu &= I_k(\mu(t_k)), \quad k \in \mathsf{N}, \ t = t_k, \\ \mu(t_0) &= \mu_0, \end{aligned} \tag{2.1}$$

where $\mu, \mu_0 \in \mathsf{R}^N$, $\Xi : \mathsf{R}_+ \times \mathsf{R}^N \to \mathsf{R}^N$, and $t_0 \in \mathsf{R}_+$. Here, $I_k : \mathsf{R}^N \to \mathsf{R}^N$ denotes the impulsive effect at moments t_k.

We analyze this system under the following assumptions:

(i) (A_0) The impulse moments satisfy $0 < t_1 < t_2 < \cdots < t_k < \ldots$ and $t_k \to \infty$ as $k \to \infty$.

(ii) The function $\Xi : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous on each interval $(t_{k-1}, t_k]$. For every $\mu \in \mathbb{R}^N$ and $k = 1, 2, \ldots$, the limit $\lim_{(t,y)\to(t_k^+,\mu)} \Xi(t, y) = \Xi(t_k^+, \mu)$ exists.

(iii) Each impulsive function $I_k : \mathbb{R}^N \to \mathbb{R}^N$ is well-defined and maps states at impulse moments.

In this paper, we assume that the function Ξ is Lipschitz continuous with respect to its second argument. Furthermore, we specify that $\Xi(t, 0) \equiv 0$ and $I_k(0) = 0$ for all k, ensuring that the trivial solution exists for system (2.1), see [22, 23]. The impulse times t_k, where $k = 1, 2, \ldots$, are fixed and satisfy $t_0 < t_1 < t_2 < \ldots$ with the property $\lim_{k\to\infty} t_k = \infty$.

The system (2.1), together with the initial condition $\mu(t_0) = \mu_0$, is assumed to have a solution $\mu(t; t_0, \mu_0) \in PC([t_0, \infty), \mathbb{R}^N)$. It is worth noting that sufficient conditions for the existence and uniqueness of global solutions to system (2.1) have been established in [16], [24], [25], [18], and [28].

The second equation in (2.1) is called the impulsive condition, and the function $I_k(\mu(t_k))$ gives the amount of jump of the solution at the point t_k. Let $V : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+^N$. Then V is said to belong to class L if,

(i) V is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^N$ and for each $\mu \in \mathbb{R}^N$, $k = 1, 2, \ldots$ and $\lim_{(t,y)\to(t_k^+,\mu)} V(t, y) = V(t_k^+, \mu)$ exists;

(ii) V is locally Lipschitz with respect to its second argument μ. For $(t, \mu) \in (t_{k-1}, t_k] \times \mathbb{R}^N$, we define the upper right Dini derivative of V with respect to (2.1) as,

$$D^+V(t, \mu) = \limsup_{h\to 0^+} \frac{1}{h}\{V(t + h, \mu + h\Xi(t, \mu)) - V(t, \mu)\} \tag{2.2}$$

Definition 2.1. *A function $\Upsilon \in C[\mathbb{R}^n, \mathbb{R}^n]$ is said to be quasi-monotone non-decreasing in μ, if $\mu \le y$ and $\mu_i = y_i$ for $1 \le i \le n$ implies $\Upsilon_i(\mu) \le \Upsilon_i(y)$, $\forall i$.*

Definition 2.2. *The set $\mu(t) \equiv 0$ in system (2.1) is defined to be:*

- *(S1) Eventually stable: For every $\epsilon > 0$, there exists a time $T = T(\epsilon) > 0$ and a corresponding $\delta = \delta(t_0, \epsilon)$ for each initial time $t_0 \in \mathbb{R}_+$ and each $\mu_0 \in \mathbb{R}^N$ such that $\|\mu_0\| < \delta$ implies $\|\mu(t; t_0, \mu_0)\| < \epsilon$ for all $t \ge t_0$.*
- *(S2) Eventually uniformly stable: The stability condition in (S1) holds with δ independent of the initial time t_0.*

Definition 2.3. *A function $a(r)$ is defined to belong to the class K if it satisfies the following conditions: $a \in C[\mathbb{R}_+, \mathbb{R}_+]$, $a(0) = 0$, and $a(r)$ is strictly increasing with respect to r.*

In this paper, we define the following sets:

$$\bar{S}_\psi = \{\mu \in \mathbb{R}^N : \|\mu\| \le \psi\}$$
$$S_\psi = \{\mu \in \mathbb{R}^N : \|\mu\| < \psi\}$$

Suffice to say here that the inequalities between vectors are understood to be component-wise inequalities.

Alongside (2.1) we shall consider a comparison system of the form

$$\begin{aligned} u' &= \Upsilon(t, u), \ t \ge t_0, \ t \ne t_k, \ k = 1, 2, \ldots \\ \Delta u &= \psi_k(u(t_k)), \ k \in N, \ t = t_k \\ u(t_0^+) &= u_0, \end{aligned} \tag{2.3}$$

existing for $t \geq t_0$, where $u \in R^n$, $R_+ = [t_0, \infty)$, $Y : R_+ \times R^n_+ \to R^n$ continuous in $(t_{k-1}, t_k]$, $\psi_k : R^n \to R^n$ and $Y(t, u)$ is quasimonotone nondecreasing in u, and $Y(t, 0) \equiv 0$, where Y is the continuous mapping of $R_+ \times R^n_+$ into R^n. The function $Y \in C[R_+ \times R^n_+, R^n]$ is such that for any initial data $(t_0, u_0) \in R_+ \times R^n$, the system (2.3) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t; t_0, u_0) \in PC([t_0, \infty), R^n)$. Note that some sufficient conditions for the existence of solution of (2.3) has been examined in [16], [18], [20] and [28].

Lemma 2.1. *Assume that conditions $A_0(i)$, (ii), (iii) hold, and that $\Xi(t, 0) = 0$ and that $I_k(0) = 0$. Then the interval J can be extended to the maximal interval of existence $[t_0, \infty)$.*

Proof. Given that conditions $A_0(i)$, (ii), and (iii) are satisfied, along with $\Xi(t, 0) = 0$ and $I_k(0) = 0$, it follows from the existence theorem for the impulsive differential equation $\mu' = \Xi(t, \mu(t))$ [18] that the solution $\mu(t) = \mu(t, t_0, \mu_0)$ of the initial value problem (2.1) is well-defined on each interval $(t_{k-1}, t_k]$, where $k = 1, 2, \ldots$.

Furthermore, since $t_0 < t_1 < \ldots$ and $\lim_{k\to\infty} t_k = \infty$, we conclude that the interval J can be extended to $[t_0, \infty)$ for $t \geq t_0$.

□

3 Main Results

In this section, we begin by proving the comparison results, then proceed to establish the necessary conditions for the uniform eventual stability of the set $x(t) \equiv 0$ of impulsive differential systems with fixed moments of impulse.

(Comparison results) Assume that

$Y \in C[R_+ \times R^n_+, R^n]$ and $Y(t, u)$ is quasimonotone nondecreasing in u for each $u \in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} Y(t, y) = Y(t_k^+, u)$ exists;

$V \in C[R_+ \times R^N, R^N_+]$ and $V \in L$ such that
$D^+V(t, \mu) \leq Y(t, V(t, \mu))$, $t \neq t_k(t, \mu) \in R_+ \times R^N$ and $V(t_k^+, \mu + I_k(\mu(t_k))) \leq \rho_k(V(t, \mu))$, $t = t_k, \mu \in S_\psi$ and the function $\rho_k : R^N_+ \to R^N_+$ is nondecreasing for $k = 1, 2, \ldots$

Let $r(t) = r(t; t_0, u_0) \in PC([t_0, T], R^n)$ be the maximal solution of the initial value problem (IVP) for the impulsive differential equation (IDE) (2.3) existing on $[t_0, \infty)$.
Then,

$$V(t, \mu(t)) \leq r(t), t \geq t_0 \tag{3.1}$$

where $\mu(t) = \mu(t; t_0, \mu_0) \in PC([t_0, T], R^N)$ is any solution of (2.1) existing on $[t_0, \infty)$, provided that

$$V(t_0^+, \mu_0) \leq u_0. \tag{3.2}$$

Proof. Let $\mu(t) = \mu(t, t_0, \mu_0)$ be any solution of (2.1) existing on $t \geq t_0$, such that $V(t_0^+, \mu_0) \leq u_0$.

Set $m(t) = V(t, \mu(t))$ for $t \neq t_k$ so that for small $h > 0$, we have

$$m(t + h) - m(t) = V(t + h, \mu(t + h)) - V(t, \mu)$$

$$m(t+h)-m(t) = V(t+h, \mu(t+h))-V(t+h, \mu(t)+h\Xi(t, \mu(t)))+V(t+h, \mu(t)+h\Xi(t, \mu(t)))-V(t, \mu)$$

Since $V(t, \mu)$ is locally Lipschitzian in μ for $t \in (t_k, t_{k+1}]$, we have

$$m(t + h) - m(t) \leq k\|\mu(t + h) - (\mu(t) + h\Xi(t, \mu(t)))\|e + V(t + h, \mu(t) + h\Xi(t, \mu(t))) - V(t, \mu)$$

Dividing by $h > 0$ and taking the limsup as $h \to 0^+$ we have

$$\limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \leq \limsup_{h\to 0^+} \frac{1}{h}[k\|\mu(t+h) - \mu(t) - h\Xi(t,\mu(t))\|e + \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, \mu(t) + h\Xi(t,\mu(t))) - V(t,\mu)]$$

where k is the local Lipschitz constant and $e = (1, 1, \ldots 1)^T$

$$D^+ m(t) = D^+ V(t, \mu(t)) \leq \Upsilon(t, V(t, \mu(t)))$$

Using condition (ii) of Theorem 3.1 we arrive at

$$V(t, \mu(t)) \leq r(t), t \neq t_k \quad (3.3)$$
$$V(t_0^+, \mu_0) \leq u_0$$

Also,

$$m(t_k^+) = V(t_k^+, \mu(t_k) + I_k(\mu(t_k^+))) \leq \psi_k(m(t_k^+)) \quad (3.4)$$

Hence, by Cor. 1.7.1 in [17], we obtain the desired estimate of 3.1. □

Corollary 3.1. *Assume that*

(i) $\Upsilon \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ *and* $\Upsilon(t,u)$ *is quasimonotone nondecreasing in* u *for each* $u \in \mathbb{R}^n$ *and* $\lim_{(t,y)\to(t_k^+,u)} \Upsilon(t,y) = \Upsilon(t_k^+, u)$ *exists;*

(ii) $V \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}_+^N]$ *and* $V \in L$ *such that* $D^+ V(t,\mu) \geq \Upsilon(t, V(t,\mu)), t \neq t_k, (t,\mu) \in \mathbb{R}_+ \times \mathbb{R}^N$ *and* $V(t, \mu + I_k(\mu(t_k))) \geq \rho_k(V(t,\mu)), t = t_k, \mu \in S_\psi$ *and the function* $\rho_k : \mathbb{R}_+^N \to \mathbb{R}_+^N$ *is nondecreasing for* $k = 1, 2, \ldots$

(iii) *Let* $p(t) = p(t; t_0, u_0) \in PC([t_0, T], \mathbb{R}_+^N)$ *be the minimal solution of the IVP for the IDE* (2.3) *existing on* $[t_0, \infty)$.
Then,

$$V(t, \mu(t)) \geq p(t), t \geq t_0 \quad (3.5)$$

where $\mu(t) = \mu(t; t_0, \mu_0) \in PC([t_0, T], \mathbb{R}^N)$ *is any solution of* (2.1) *existing on* $[t_0, \infty)$, *provided that*

$$V(t_0^+, \mu_0) \geq u_0. \quad (3.6)$$

Proof. Let $\mu(t) = \mu(t, t_0, \mu_0)$ be any solution of (2.1) existing on $t \geq t_0$, such that $V(t_0^+, \mu_0) \geq u_0$.

Set $m(t) = V(t, \mu(t))$ for $t \neq t_k$ so that for small $h > 0$, we have

$$m(t+h) - m(t) = V(t+h, \mu(t+h)) - V(t,\mu)$$

$$m(t+h) - m(t) = V(t+h, \mu(t+h)) - V(t+h, \mu(t) + h\Xi(t,\mu(t))) + V(t+h, \mu(t) + h\Xi(t,\mu(t))) - V(t,\mu)$$

Since $V(t,\mu)$ is locally Lipschitzian in μ for $t \in (t_k, t_{k+1}]$, we have

$$m(t+h) - m(t) \geq k\|\mu(t+h) - (\mu(t) + h\Xi(t,\mu(t)))\|e + V(t+h, \mu(t) + h\Xi(t,\mu(t))) - V(t,\mu)$$

Dividing by $h > 0$ and taking the limsup as $h \to 0^+$ we have

$$\limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \geq \limsup_{h\to 0^+} \frac{1}{h}[k\|\mu(t+h) - \mu(t) - h\Xi(t,\mu(t))\|e + \limsup_{h\to 0^+} \frac{1}{h}$$

$-[V(t+h, \mu(t)+h\Xi(t, \mu(t))) - V(t, \mu)]\, h$

where k is the local Lipschitz constant and $e = (1, 1, \ldots 1)^T$

$$D^+m(t) = D^+V(t, \mu(t))$$

Using condition (ii) of Cor 3.2 we arrive at

$$D^+V(t, \mu) \geq \Upsilon(t, m(t)), t \neq t_k m(t_0^+) \geq u_0 \tag{3.7}$$

Also,

$$m(t_k^+) = V(t_k^+, \mu(t_k) + I_k(\mu(t_k^+))) \leq \psi_k(m(t_k^+)) \tag{3.8}$$

Hence, by Cor. 1.7.1 in [17], we obtain the desired estimate of 3.5. □

[Uniform Stability] Assume the following

(i) $\Upsilon \in C[\mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n]$ satisfies $(A_0)(ii)$ and $\Upsilon(t, u)$ is quasi-monotone non-decreasing in u with $\Upsilon(t, 0) \equiv 0$.

(ii) $V : \mathbb{R}_+ \times S_\psi \to \mathbb{R}^N_+ V \in L$ is locally Lipschitzian in μ with $V(t, 0) \equiv 0$ such that

$$D^+V(t, \mu) \leq \Upsilon(t, V(t, \mu)), t \neq t_k, (t, \mu) \in \mathbb{R}_+ \times S_\psi \tag{3.9}$$

holds for all $(t, \mu) \in \mathbb{R}_+ \times S_\psi$.

(iii) there exists a $\psi_0 > 0$ such that $\mu_0 \in S_\psi$ implies that

$$\mu + I_k(\mu) \in S_\psi \text{ and } V(t_k^+, \mu + I_k(\mu(t_k))) \leq \psi_k(V(t, \mu)), t = t_k, \mu \in S_\psi$$

and the function $\psi_k : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ is nondecreasing for $k = 1, 2, \ldots$

(iv) $b(\|\mu\|) \leq V_0(t, \mu) \leq a(\|\mu\|)$, where $a, b \in \mathsf{K}$ and $V_0(t, \mu) = \sum_{i=1}^N V_i(t, \mu)$

Then the uniform eventual stability of the set $u(t) \equiv 0$ of the IDE (2.3) implies the uniform eventual stability of the set $\mu(t) \equiv 0$ of (2.1).

Proof. Let $0 < \epsilon < \psi$ and $t_0 \in \mathbb{R}_+$ be given.

Assume that the solution $u = 0$ of (3.5) is uniformly stable. Then, for each $b(\epsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a positive function $\delta_1 = \delta_1(\epsilon) > 0$ such that, whenever

$$u_0 = \sum_{i=1}^n u_{i0} \leq \delta, \text{ we have } \sum_{i=1}^n u_i(t; t_0, u_0) < b(\epsilon), \quad t \geq t_0, \tag{3.10}$$

where $u(t; t_0, u_0)$ represents any solution of (3.5).

Let us choose $V(t_0^+, \mu_0) \leq u_0$ and

$$\sum_{i=1}^{\infty} u_{i0} = a(t_0, \|\mu_0\|)$$

Since $a(t, \mathsf{K})$ and $a \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ we can find a positive function $\delta = \delta(t_0, \epsilon) > 0$ such that

$$a(t_0, \|\mu_0\|) < \delta_1 \text{ and } \|\mu_0\| < \delta \tag{3.11}$$

hold simultaneously. We claim that if

$$\|\mu_0\| \leq \delta, \text{ then } \|\mu(t, t_0, \mu_0)\| \leq \epsilon, t \geq t_0.$$

Assume, for contradiction, that this claim is false. Then, there would exist a point $t_1 > t_0$ and a solution $\mu(t)$ with $\|\mu_0\| < \delta$ such that

$$\|\mu(t_1)\| = \epsilon \text{ and } \|\mu(t)\| < \epsilon, \text{ for } t \in [t_0, t_1). \tag{3.12}$$

This implies that $\mu(t) + I_k(\mu(t_k)) \in S_\psi$ for $t \in [t_0, t_1)$.
From equation (3.1) we have that

$$V_0(t, \mu(t)) \leq r_0(t, t_0, u_0) \text{ for } t \in [t_0, t_1). \tag{3.13}$$

Combining condition (*iv*) and (3.13) we have

$$b(\epsilon) \leq \sum_{i=1}^{n} V_i(t_1, \mu(t_1)) \leq \sum_{i=1}^{n} r_i(t; t_0, u_0) \tag{3.14}$$

Using equations (3.14), (3.13) and (3.10) we have,

$$b(\epsilon) \leq \sum_{i=1}^{n} V_i(t_1, \mu(t_1)) \leq \sum_{i=1}^{n} r_i(t; t_0, u_0) < b(\epsilon)$$

which leads to an absurdity that $b(\epsilon) < b(\epsilon)$.
Hence, the uniform eventual stability of the set $u(t) = 0$ of (2.3) implies the uniform eventual stability of $\mu(t) = 0$ of (2.1). □

4 Application

Let the points t_k, $t_k < t_{k+1}$, $lim_{k\to\infty} t_k \to \infty$ be fixed. Consider the vector impulsive differential equations

$$\begin{aligned} x_1' &= -4x_1 + x_2 sin x_1 + x_1 sec x_2, \ t \neq t_k \\ x_2' &= x_1 cos x_2 - 2x_2 sec x_1 - x_2 sin x_1, \ t \neq t_k \\ \Delta x_1 &= c_k, \ \Delta x_2 = d_k, \ k = 1, 2, .. \end{aligned} \tag{4.1}$$

for $t \geq t_0$, with initial conditions

$$x_1(t_0^+) = x_0 \quad \text{and} \quad x_2(t_0^+) = x_0$$

Consider a vector $V = (V_1, V_2)^T$, where
$V_1(t, x_1, x_2) = |x_1|$ and $V_2(t, x_1, x_2) = |x_2|$, with $x = (x_1, x_2) \in \mathbb{R}^2$, and its associated norm defined by $\|x\| = |x_1| + |x_2|$.
Now

$$V_0(t, x) = \sum_{i=1}^{2} V_i(t, x_1, x_2) = |x_1| + |x_2| \tag{4.2}$$

and so $b(\|x\|) \leq V_0(t, x) \leq a(\|x\|)$ with $b(r) = r$ and $a(r) = r^2$, implying that $a, b \in \mathrm{K}$. From (2.2) we compute the Dini derivative for $V_1 = |x_1|$ for $t > 0, t \neq t_k$ as follows:

$$D^+V_1(t; x_1, x_2) = \limsup_{h\to 0^+} \frac{1}{h} \{ V(t + h, x + hf(t, x)) - V(t, x) \}$$

$$D^+V_1 = \limsup_{h\to 0^+} \frac{1}{h} \{ |x_1 + hf_1(t, x_1)| - |x_1| \} \tag{4.3}$$

$$= \limsup_{h\to 0^+} \frac{1}{h} \; hf_1(t, x_1)\Big\}$$

$$\leq \; f_1(t, x_1)$$

$$= \; -4x_1 + x_2 sinx_1 + x_1 secx_2$$

$$= \; x_1(-4 + secx_2) + x_2 sinx_1$$

$$\leq \; |x_1|(-4 + \frac{1}{|cosx_2|}) + |x_2|(|sinx_1|)$$

where trigonometric functions are bounded by 1

$$\leq \; |x_1|(-4+1) + |x_2|(1)$$

$$D^+V_1 \; \leq \; -3V_1 + V_2 \tag{4.4}$$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..$, we have

$$V(t, x(t) + c_k) = |c_k + x(t)| \leq V(t, x(t))$$

Again, for $V_2 = |x_2|$, *wehave*

$$D^+V_2 \; \leq \; f_2(t, x_2)$$

$$= \; x_1 cosx_2 - 2x_2 secx_1 - x_2 sinx_1$$

$$= \; x_1 cosx_2 - x_2(2secx_1 - sinx_1)$$

$$\leq \; |x_1|(|cosx_2|) - |x_2|(\frac{2}{|cosx_1|} + |sinx_1|)$$

$$\leq \; |x_1| - |x_2|(2+1)$$

$$D^+V_2 \; \leq \; V_1 - 3V_2 \tag{4.5}$$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..$, we have

$$V(t, x(t) + d_k) = |d_k + x(t)| \leq V(t, x(t))$$

Combining (4.5) and (4.4) we have

$$D^+V \leq \begin{matrix} -3 & 1 \\ 1 & -3 \end{matrix} \begin{matrix} V_1 \\ V_2 \end{matrix} = g(t, V) \tag{4.6}$$

Now consider the comparison system

$$u' = g(t, u) = Au \tag{4.7}$$

where $A = \begin{matrix} -3 & 1 \\ 1 & -3 \end{matrix}$.

The vectorial inequality (4.7) and all other conditions of Theorem (3) are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (4.1) is uniformly eventually stable . Therefore, the set $x(t) \equiv 0$ is uniformly eventually stable.

5 Conclusion

This work investigates the uniform eventual stability of an impulsive differential system using a class of piecewise continuous auxilliary functions that generalise the vector Lyapunov functions. We provide necessary requirements for the uniform eventual stability solution with an illustrated case and comparing findings.

References

[1] Lakshmikantham, V., Bainov, D. D., and Simeonov, P. S. worked on this in 1989. Fundamentals of Impulsive Differential Equations. Modern Applied Mathematics Series, volume 6, issue 6.
In 2024, Achuobi, J.O., Akpan, E.P., George, R., and Ofem, A.E. published a document. Caputo fractional time-dependent systems with delay and stability analysis using vector lyapunov functions, AIMS Mathematics. [3] Adeyanju, A. A., Adams, D. O., Oni, S. A., & Olutimo, A. L. (2023). Mathematical Perspectives, 9, 28079–28099. doi: 10.3934/math.20241362. With respect to the asymptotic behaviour, boundedness, and stability of solutions to third-order differential equations. Chapter 9: Optimisation Theory and Applications published in the International Journal of Mathematical Sciences, volume 9, issue 2, pages 121–132. The citation is from the following source: [4] Ante et al. (2024): https://doi.org/10.6084/zenodo.10384294. In the African Journal of Mathematics and Statistics Studies, the authors discuss a new auxiliary Lyapunov function and the uniform asymptotic practical stability of nonlinear impulsive Caputo fractional differential equations using a newly modelled generalised Dini derivative. verses 7:11–33.
With reference to [5] Ante et al. (2024), Atsu et al., Abraham et al., Itam et al., Oduobuk et al., and Inyang et al. The asymptotic practical stability of nonlinear impulsive Caputo fractional differential equations, as well as a class of piecewise continuous Lyapunov functions, are addressed in a recent publication of the IEE-SEM Journal, with reference number 12:2320-9151.
In 2024, Ante, Atsu, Maharaj, Abraham, and Narain published a study. Regarding a set of piecewise continuous Lyapunov functions and the uniform eventual stability of nonlinear impulsive Caputo fractional differential equations via a newly modelled generalised Dini derivative, there is Asia Pac. Mathematics, volume 11, issue 99, pages 1-20. APA reference: [7] Ante et al. (2024). https://apjm.apacific.org/PDFs/11-99.pdf. Regarding nonlinear impulsive differential equations' vector Lyapunov functions and eventual stability, the article may be found in the International Journal of Mathematical Analysis and Modelling, volume 7, issue 2, pages 185–199.
In a study conducted by Atsu, Ante, Inyang, and Akpan in 2024, the authors cited[8] their work. Using a newly modelled generalised Dini derivative, we explore vector Lyapunov functions and actual stability of nonlinear impulsive Caputo fractional differential equations. 20:28- IOSR Journal of Mathematics