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Transmission of the Zeghdoudi Distribution: Concepts and Problems

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Abstract- The TZD, or Transmuted Zeghdoudi distribution, is obtained by use of the QRTM method. Derived mathematical features of the suggested distribution include, among other things, moments, skewness, kurtosis, mean residual lifespan, hazard rate, order statistics, reliability analysis, and so on. In order to get parameter estimates, estimation properties such as the maximum likelihood estimation process are developed. Data sets are examined to demonstrate its usefulness and to compare its performance to those of other distributions.

Keywords—*Zeghdoudi Distribution, Quadratic Rank Transmutation Map, Moment Generating Function, Order Statistics, Maximum Likelihood Estimation*

I. Introduction

Prediction and analysis of real-life data rely heavily on certain statistical distributions. Financial, biological, physical, actuarial, and engineering data are all used to make predictions and analyses using the statistical distribution. Lifetime data sets have been modelled using many popular continuous distributions, including Exponential, Gamma, Lindley, and Weibull. The area of data modelling has benefited greatly from the introduction of several different distributions within the last 30 years. Modern technology allows for the collection of ever-increasing amounts of data from a wide variety of sources. Therefore, a new distribution will be developed via study to provide much more accurate estimations by fitting the data better. To better fit the lifetime data and have high flexibility, some one parameter mixture distributions were used, such as Beta-Pareto[1], Lindley[2], Beta-Laplace[3], Xgamma[4], Weibull-Pareto[5], Beta Exponentiated-Pareto[6], Transmuted Pranav Distribution[7], Transmuted Lindley Distribution[8], Gamma-Lindley Distribution[9], along with others. When it comes to using probability distributions to describe phenomena in the actual world, the exponential families of distributions are essential.

Recently, on parameter exponential families of distribution which is the mixture of Gamma(2,θ) and Gamma(3,θ) with mixing probability $\theta/(2+\theta)$ have been introduced and named as Zeghdoudi distribution[10]. The probability density function (pdf) of Zeghdoudi distribution is defined as

$$g(x;\theta) = \frac{\theta^3 x(1+x)}{2+\theta} e^{-\theta x} \quad x > 0, \theta > 0 \tag{1}$$

and its cumulative distribution function (cdf) is given by

$$G(x) = 1 - \frac{\theta^2 x^2 + \theta(\theta+2)x + \theta + 2}{2+\theta} e^{-\theta x} \quad x > 0, \theta > 0. \tag{2}$$

We use the transmutation map approach which was suggested by Shaw et. al.[11]

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x), \ \ |\lambda| \leq 1, \tag{3}$$

where $G(x)$ is the cdf of the base distribution.

When $\lambda = 0$ we have the distribution of the Zeghdoudi random variable. In this paper, we have introduced a transmuted version of the Zeghdoudi Distribution using Quadratic Rank Transmutation Map (QRTM) approach. The uses of this transmutation approach have played a vital role in modeling the lifetime data. We named the distribution as Transmuted Zeghdoudi Distribution (TZD).

The paper is organized as follows, Section II contains the introduction of pdf and cdf of Transmuted Zeghdoudi Distribution. Section III contains the mathematical properties of the distribution such as moments, moment generating function and second, third and fourth order moment. Section IV contains the measures of skewness and kurtosis and plotted for different values. Section V contains the order statistics. Section VI contains Reliability analysis and Hazard

Function, Section VII contains simulation and parameter estimation using Maximum Likelihood Estimation., Section VIII contains the application to two real lifetime data and compare with two other distributions using related measures. Section IX concludes research work.

II. Transmuted Zeghdoudi distribution

Definition 4.1 *A random variable X is said to have the transmuted Zeghdoudi distribution with parameter θ and λ if its probability density is defined as*

$$f(x) = \frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta}[(1-\lambda) + 2\lambda(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1)e^{-\theta x}] \quad ; \quad \theta, x > 0, \tag{4}$$

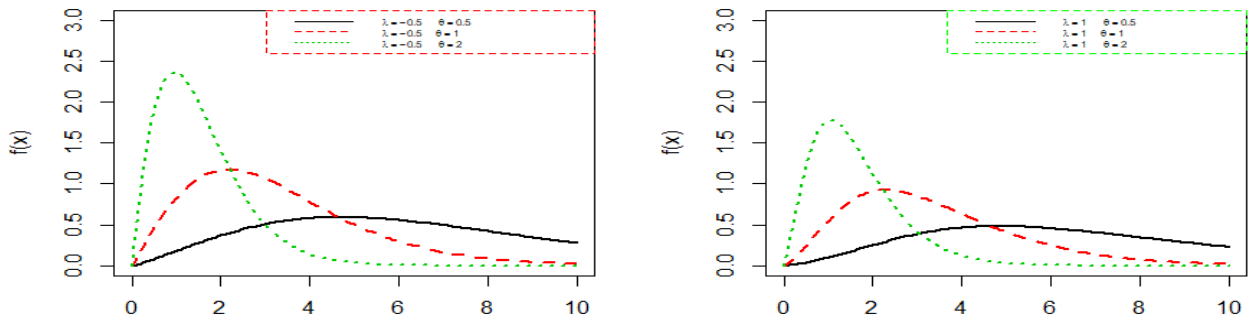


Figure 1: Plot of p.d.f of Transmuted Zeghdoudi Distribution

and its cdf is given by

$$F(x) = [1 - (\frac{\theta^2 x^2}{2+\theta} + \theta x + 1)\, e^{-\theta x}]\,[1 + \lambda\,(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1)\, e^{-\theta x}] \quad ; \quad \theta, x > 0. \tag{5}$$

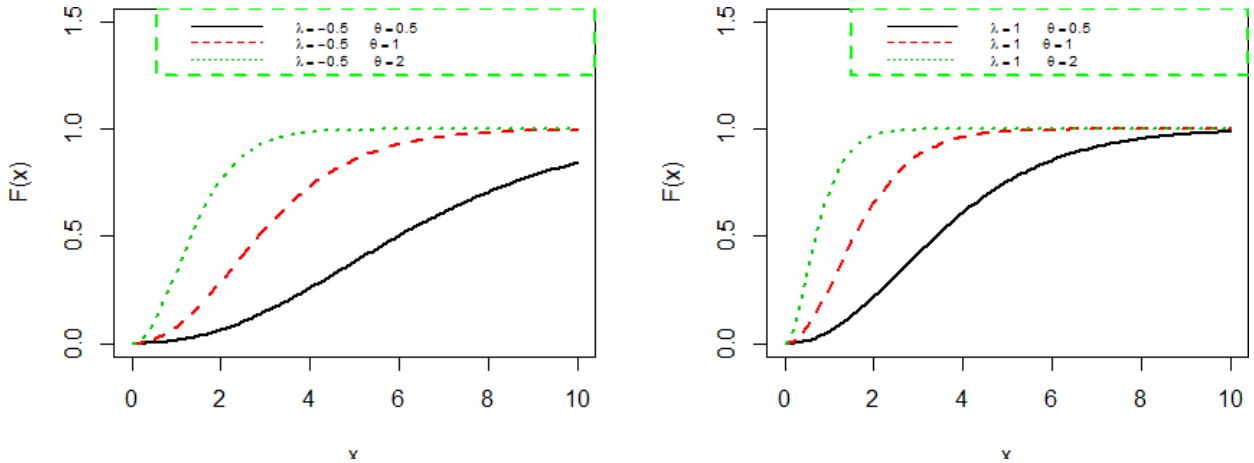


Figure 2: Plot of c.d.f. of Transmuted Zeghdoudi Distribution

The transmuted Zeghdoudi distribution is an extended model of Gamma distribution to analyse the more complex lifetime data.

III. Moments and quantiles

Let x ~TG(θ, λ), then the r^{th} raw moment about the origin is given by

$$E(X^k) = \int_0^\infty x^k \frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta}[(1-\lambda) + 2\lambda(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1)e^{-\theta x}]\, dx. \tag{6}$$

And the first, second, third and fourth order moment about origin are given by:

$$\mu_1' = \frac{\theta^3}{2+\theta}\cdot\left[\frac{2}{\theta^3}\left(1+\frac{3}{\theta}\right) - \frac{2\lambda}{\theta^3}\left(1+\frac{3}{\theta}\right) + \frac{48\lambda\theta^2}{(\theta+2)(2\theta)^5}\left(1+\frac{5}{2\theta}\right) + \frac{12\lambda\theta}{(2\theta)^4}\left(1+\frac{2}{\theta}\right) + \frac{4\lambda}{(2\theta)^3}\left(1+\frac{3}{2\theta}\right)\right],$$

$$\mu_2' = \frac{\theta^3}{2+\theta}\cdot\left[\frac{6}{\theta^4}\left(1+\frac{4}{\theta}\right) - \frac{6\lambda}{\theta^4}\left(1+\frac{4}{\theta}\right) + \frac{240\lambda\theta^2}{(\theta+2)(2\theta)^6}\left(1+\frac{3}{\theta}\right) + \frac{48\lambda\theta}{(2\theta)^5}\left(1+\frac{5}{2\theta}\right) + \frac{12\lambda}{(2\theta)^4}\left(1+\frac{2}{\theta}\right)\right],$$

$$\mu_3' = \frac{\theta^3}{2+\theta}\cdot\left[\frac{24}{\theta^5}\left(1+\frac{5}{\theta}\right) - \frac{24\lambda}{\theta^5}\left(1+\frac{5}{\theta}\right) + \frac{1440\lambda\theta^2}{(\theta+2)(2\theta)^7}\left(1+\frac{7}{2\theta}\right) + \frac{240\lambda\theta}{(2\theta)^6}\left(1+\frac{3}{\theta}\right) + \frac{48\lambda}{(2\theta)^5}\left(1+\frac{5}{2\theta}\right)\right],$$

$$\mu_4' = \frac{\theta^3}{2+\theta}\cdot\left[\frac{120}{\theta^6}\left(1+\frac{6}{\theta}\right) - \frac{120\lambda}{\theta^6}\left(1+\frac{6}{\theta}\right) + \frac{10080\lambda\theta^2}{(\theta+2)(2\theta)^8}\left(1+\frac{4}{\theta}\right) + \frac{1400\lambda\theta}{(2\theta)^7}\left(1+\frac{7}{2\theta}\right) + \frac{240\lambda}{(2\theta)^6}\left(1+\frac{3}{\theta}\right)\right].$$

As variance, skewness & kurtosis calculative expressions are so complicate, therefore the graph of skewness & kurtosis is plotted.

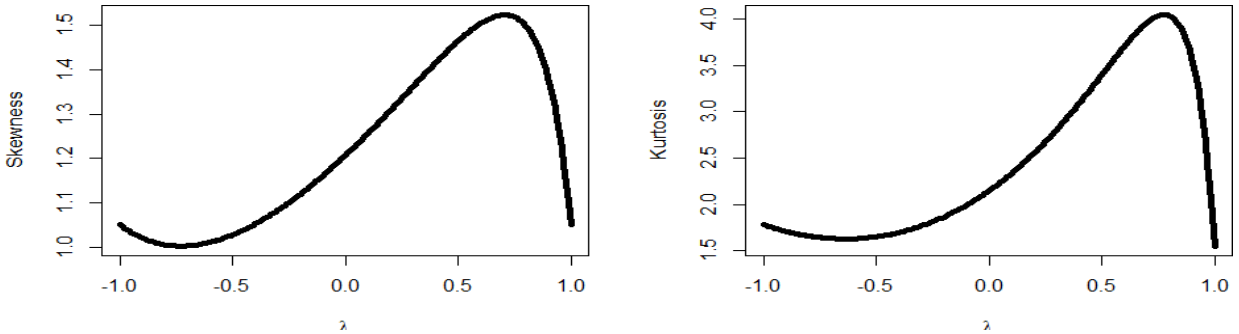


Figure 3: The graph of skewness and kurtosis for λ from -1 to +1.

IV. Moment Generating Function

The moment generating function in terms of the moment are given by

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}\frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta}[(1-\lambda)+2\lambda(\frac{\theta^2x^2}{2+\theta}+\theta x+1)e^{-\theta x}]\,dx$$
$$= \frac{\theta^3}{2+\theta}\left[\left(\frac{1-\lambda}{(\theta-t)^2}\right)\left(1+\frac{2}{\theta-t}\right) + \frac{12\lambda\theta^2}{(2+\theta)(2\theta-t)^4}\left(1+\frac{4}{2\theta-t}\right)\right.$$
$$\left. + \frac{4\lambda\theta}{(2\theta-t)^3}\left(1+\frac{3}{2\theta-t}\right) + \frac{2\lambda}{(2\theta-t)^2}\left(1+\frac{2}{(2\theta-t)^3}\right)\right].$$

V. Order Statistics

Let $Y_{(1)}, Y_{(2)}, .., Y_{(n)}$ be the ordered random sample drawn from the Transmuted Zeghdoudi Distribution, then the c.d.f. and p.d.f. of the r^{th} order statistics is given by

$$G_r(y) = \sum_{i=r}^{n} \binom{n}{i} [G(y;\theta;\lambda)]^i[1-G(y;\theta;\lambda)]^{n-i},$$

$$g_r(y) = \frac{n!}{(r-1)!(n-r)!}[G(y;\theta;\lambda)]^{r-1}[1-G(y;\theta;\lambda)]^{n-r}g(y;\theta;\lambda).$$

So, the p.d.f. and c.d.f. of $y_{(1)}$ and $y_{(n)}$ is given by

$$g_{y_1}(y) = n\left[\frac{\theta^3x(1+x)e^{-\theta x}}{2+\theta}((1-\lambda)+2\lambda(\frac{\theta^2x^2}{2+\theta}+\theta x+1)e^{-\theta x})\right]\left[1-(\frac{\theta^2x^2}{\theta+2}+\theta x+1)\,e^{-\theta x}\right]\left[1+\lambda\,(\frac{\theta^2x^2}{\theta+2}+\theta x+1)\,e^{-\theta x}\right]^{n-1},$$

$$g_{y_n}(y) = n\left[\frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta}((1-\lambda)+2\lambda(\frac{\theta^2x^2}{2+\theta}+\theta x+1)e^{-\theta x})\right]\left[1-(\frac{\theta^2x^2}{\theta+2}+\theta x+1)\,e^{-\theta x}\right]\left[1+\lambda\,(\frac{\theta^2x^2}{\theta+2}+\theta x+1)\,e^{-\theta x}\right]^{n-1},$$

$$G_{y_1}(y) = 1-([1-(\frac{\theta^2x^2}{2+\theta}+\theta x+1)\,e^{-\theta x}]\,[1+\lambda\,(\frac{\theta^2x^2}{2+\theta}+\theta x+1)\,e^{-\theta x}])^{n-1},$$

$$G_{y_n}(y) = ([1-(\frac{\theta^2x^2}{2+\theta}+\theta x+1)\,e^{-\theta x}]\,[1+\lambda\,(\frac{\theta^2x^2}{2+\theta}+\theta x+1)\,e^{-\theta x}])^{n}.$$

VI. Survival Properties

The hazard rate function or failure rate function for a continuous distribution with pdf f(x), cdf F(x) and survival function (sf) S(x) is defined as,

$$h(x) = \ln\lim_{\Delta x\to 0}\frac{P(X<x+\Delta x|X>x)}{\Delta x} = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)}$$

For the transmuted zeghdoudi distribution, the hazard rate function is given by

$$h(x) = \frac{\frac{\theta^3x(1+x)e^{-\theta x}}{2+\theta}[(1-\lambda)+2\lambda(\frac{\theta^2x^2}{2+\theta}+\theta x+1)e^{-\theta x}]}{1-[1-(\frac{\theta^2x^2}{2+\theta}+\theta x+1)\,e^{-\theta x}]\,[1+\lambda\,(\frac{\theta^2x^2}{2+\theta}+\theta x+1)\,e^{-\theta x}]}$$

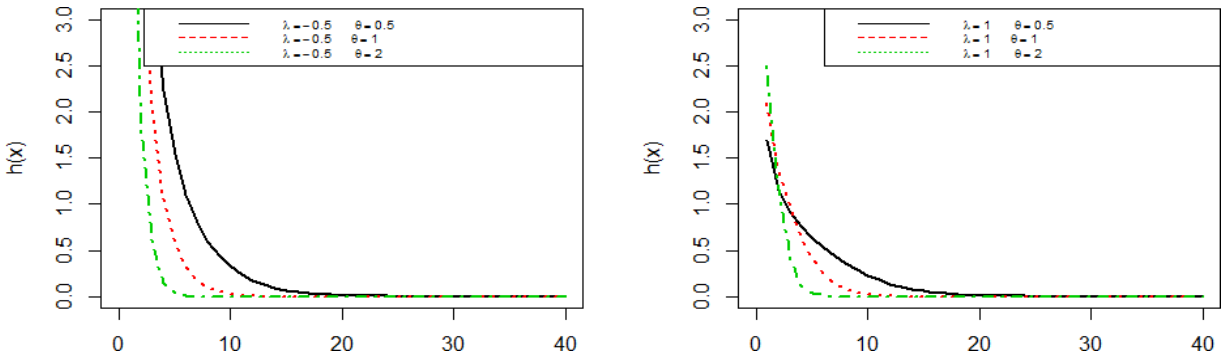


Figure 4: Plot of hazard function of Transmuted zeghdoudi distribution.

VII. Estimation of the parameters

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from the pdf in (4). The log likelihood equation is as follows

$$L(\theta,\lambda) = 3nlog(\theta) - nlog(2+\theta) + \sum_i^n log\left[x_i(x_i+1)\right] - n\theta\bar{x} + \sum_i^n log\,(1-\lambda+2\lambda\,(\frac{\theta^2x_i^2}{\theta+2}+\theta x+1)\,e^{-\theta x}). \quad (7)$$

To get the maximum likelihood estimator of λ and θ, we have to solve $\frac{\partial L(\theta,\lambda)}{\partial\theta} = 0$ and $\frac{\partial L(\theta,\lambda)}{\partial\lambda} = 0$ Hence, we have

$$\frac{3n}{\theta} - \frac{n}{2+\theta} - n\bar{x} + 2\lambda\sum_{i=1}^{n}\frac{\frac{(x_i+2\theta(\theta+2)x_i+\theta^2x_i)e^{-\theta x_i}}{(\theta+2)^2} - x_i\,(\frac{\theta^2x_i^2}{\theta+2}+\theta x_i+1)\,e^{-\theta x_i}}{(1-\lambda+2\lambda\,(\frac{\theta^2x_i^2}{2+\theta}+\theta x_i+1)\,e^{-\theta x_i})} = 0 \quad (8)$$

and

$$\sum_{i=1}^{n} \left(\frac{2\left(\frac{\theta^2 x_i^2}{2+\theta} + \theta x_i + 1\right) e^{-\theta x_i} - 1}{1 - \lambda + 2\lambda\left(\frac{\theta^2 x_i^2}{\theta+2} + \theta x_i + 1\right) e^{-\theta x_i}} \right) = 0 \quad (9)$$

.

To get the MLE of λ and θ, we use the R- statistical package to solve these equation numerically.

VIII. Simulation Study

To generate the random sample from transmuted zeghdoudi distribution, we proceed as follows:
Let $F(x) = p$, where p follows $uniform(0,1)$.
i.e.

$$(1+\lambda)G(x) - \lambda G^2(x) = p$$

i.e.

$$U = G(x) = \frac{(1+\lambda) \pm \sqrt{(1+\lambda^2 - 4\lambda p)}}{2\lambda}, |\lambda| \leq 1$$

We will choose that solution which lies in $(0,1)$. Here $G(x)$ is the cdf of zeghdoudi random variable. Then, we follow the approach of generating sample from zeghdoudi distribution. Hence the algorithm is

- Generate $p_i \sim uniform(0,1), i = 1,2,\dots,n.$
- Calculate $U_i = \frac{(1+\lambda)+\sqrt{(1+\lambda^2-4\lambda p}_i}{2\lambda}, i = 1,2,\dots,n.$
- Generate $V_i \sim gamma(2,\theta), i = 1,2,\dots,n.$
- Generate $W_i \sim gamma(3,\theta), i = 1,2,\dots,n.$
- If $U_i \leq \frac{\theta}{2+\theta}$, then set $X_i = V_i$. Otherwise, set $X_i = W_i$.

A Monte Carlo simulation study is carried out considering $N = 10000$ times for different values of n and presented graphically for $\theta = 0.8$ and $\lambda = 0.6$ (Figure). The following two measures were computed

- Average bias of the simulated estimates $\hat{\theta}_i, i = 1,2,\dots,N$: $\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)$
- Average Mean Square Error (MSE) of the simulated estimates $\hat{\theta}_i, i = 1,2,\dots,N$: $\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2$

IX. Data Analysis

In this section,we use a real lifetime data and shown that the flexibility of Transmuted Zeghdoudi Distribution.

Data set 1- represent the waiting times (in mins) before customer service in bank 0.8 0.8 1.3 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19, 19.9, 20.6, 21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5

Table 1: The MLEs of parameter(s), negative log-likelihood and AIC values for two survival model.

Survival model	MLE	negative log-likelihood	AIC -2log-likelihood+2K ,K=parameter
Transmuted Zeghdoudi(θ,)	$\hat{\theta} = 0.2593$, $\hat{\lambda} = 0.3784$	314.271	632.542
Zeghdoudi(θ)	$\hat{\theta} = 0.3886$	334.234	670.468
Lindley(θ)	$\hat{\theta} = 0.1865$	319.0468	640.093
Exponential(θ)	$\hat{\theta} = 0.1012$	329.047	660.094
Xgamma(θ)	$\hat{\theta} = 0.2633$	332.237	666.474

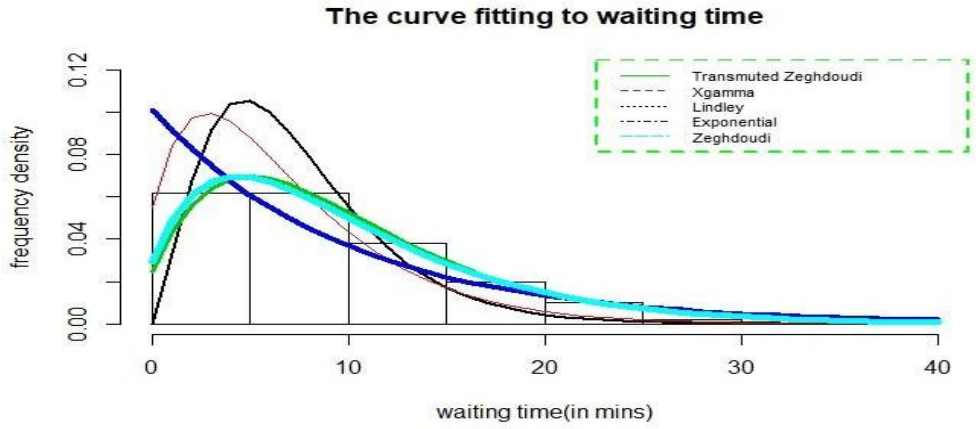


Figure 5: The graph showing fitting of the different distribution to the waiting time(in mins) before the customer service.

X. Conclusion

An expansion of the one-parameter Zeghdoudi Distribution, we provide a novel two-parameter Transmuted Zeghdoudi Distribution (TZD) in this paper. We used real-life data sets, estimated parameters, and obtained various statistical features. This research presents the novel distribution and demonstrates that it outperforms existing models by applying it to the "waiting times before customer service in bank" data. We hope that many researchers will use this page as a reference for their own work in this area.

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"Emerging Trends in Data Mining- An Algorithms used, Challenges and its Significance in Current Scenario," published in the International Journal of Computer Sciences and Engineering in 2020, is authored by Gagan Gurung, Rahul Shah, and Dhiraj Prasad Jaiswal and can be found on pages 78–82.
In their 2020 article titled "A Differential Computational Encryption Modelling Technique on Textual Data," the authors Logunleko, Odufowora, and Gbolagade discuss a computational encryption method. The article is published in the International Journal of Scientific Research in Computer Science and Engineering, volume 8, issue 1, pages 81–86.