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Experimental Distribution with Beta Weights: Theory and Practice

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Abstract
[10] developed the weighted exponential distribution by expanding on the concept of [2], which included using an exponential model with a form parameter. In order to create the beta weighted exponential distribution, which is based on the logit of beta function [12], we used two shape factors to modify the current weighted exponential distribution. The new distribution's statistical features were investigated. The R software code was used to estimate the parameters using the maximum likelihood estimation approach. We then compared the novel distribution's features to those of well-known distributions using a data set that included the survival times of guinea pigs injected with varying amounts of tubercle bacilli. Based on our analysis, the new distribution is far more adaptable and flexible.

1 Introduction

It is not novel to include form parameter(s) into an exponential model, and there are several approaches to do so. Many studies have relied on the method first proposed by [12], which makes use of the logit of the beta distribution. Similarly to how [14] used the logit of the Beta distribution, it extended the exponential distribution. In order to provide an alternative to the exponential and Weibull distributions, [9] explored the generalised exponential distribution.

[2] initiated a method of obtaining weighted distributions from independently identically distributed (*i.i.d.*) random variables Y_1 and Y_2 based on the expression

$$f(y) = \frac{1}{P(\alpha X_1 > X_2)} f_F(y) F_F(\alpha y),\ \alpha > 0 \tag{1.1}$$

where f(y) and F(y) were the *pdf* and *cdf* of Y respectively and α was an unknown parameter. [10] slightly modified Azzalini's approach to obtain the

Weighted Exponential (WE) distribution defined as

$$f_F(y) = \frac{1}{P(\alpha^{\frac{1}{\beta}} F_1 > F_2)} f_F(y) F_F(\alpha^{\frac{1}{\beta}} y),\ \alpha, \beta > 0 \tag{1.2}$$

where α and β were unknown parameters.

2 Material and Method

2.1 The Beta Weighted exponential (BWE) Distribution

2.2 The Probability Density Function (pdf) of BWE Distribution

We upgraded the model by [10] using the logit of beta link function method of [12] that has been used extensively in literature. [8] investigated on beta Weibull distribution, [1] proposed the beta pareto distribution, [4] studied some statistical properties of exponentiated weighted Weibull distribution, [3] also analyzed life length of components estimates with beta-weighted Weibull distribution among others; they all used the method of logit of beta link function in their work. The link function is given as

$$g(y) = \frac{1}{B(a,b)}[F(y)]^{a-1}[1 - F(y)]^{b-1}f(y),\ a, b\ >\ 0 \qquad (2.2.1)$$

where

$$f(y) = \frac{\lambda(1+\alpha)}{\alpha}[exp(-\lambda y)\ (1 - exp(-\alpha\lambda y)]$$

$$F(y) = \frac{[(1+\alpha)(1 - exp(-\lambda y) + exp(-\lambda y(1+\alpha)) - 1]}{\alpha}$$

where,

α,λ and $y > 0$ are shape and scale parameter; $f(y)$ *and* $F(y)$ are *pdf* and *cdf* respectively of parent distribution, i.e. weighed exponential distribution

We substitute the *pdf* $f(y)$ and *cdf* $F(y)$ in (2.2.1) to obtain the pdf of beta weighted exponential probability density function as

$$f_{BWE}(y) = \frac{1}{B(a,\ b)}[\frac{[(1+\alpha)(1 - exp(-\lambda y) + exp(-\lambda y(1+\alpha)) - 1]}{\alpha}]^{a-1}$$

$$[1 - \frac{[(1+\alpha)(1 - exp(-\lambda y)) + exp(-\lambda y(1+\alpha)) - 1]}{\alpha}]^{b-1}$$

$$[\frac{\lambda(1+\alpha)\ exp(-\lambda y)\ (1 - exp(-\lambda\alpha y))}{\alpha}]$$

$$a, b, \lambda, \alpha, y\ \approx BWE(\alpha, \lambda, a, b, y > 0) \qquad (2.2.2)$$

where α was shape parameter, λ was scale parameter, *a and b* were additional shape parameters. Expression (2.2.2) is the *pdf* of the proposed BWE distribution.

We set

$$W(y) = F(y) = \frac{[(1+\alpha)(1-exp(-\lambda y)) + exp(-\lambda y(1+\alpha)) - 1]}{\alpha}$$

to obtain

$$\frac{dW}{dy} = \frac{\lambda(1+\alpha)[exp(-\lambda\ )(1-exp(-\alpha\lambda y)]}{\alpha}[\ \frac{(1+\alpha)(1-exp(-\lambda y))+exp(-\lambda y(1+\alpha))-1]}{\alpha}[\quad (2.2.3)$$

Then expression (2.2.2) becomes

$$f_{BWE}(y) = \frac{1}{B(a,b)}[W(y)]^{a-1}[1 - W(y)]^{b-1}dW(y) \quad (2.2.4)$$

Figure 1 to 5 below show the PDF of BWE at values of a=100,50,20,10,5; b=6,5,3,2.5,2 ; λ=1.5; and α =2

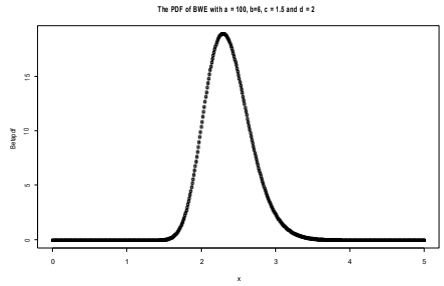


Fig 1. Plot of PDF of BWE at (a,b) = (100,6)

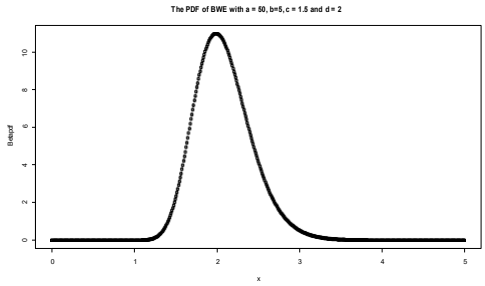
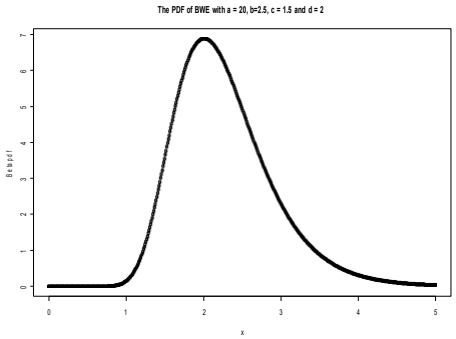


Fig 2. Plot of PDF of BWE at (a,b) = (50,5)



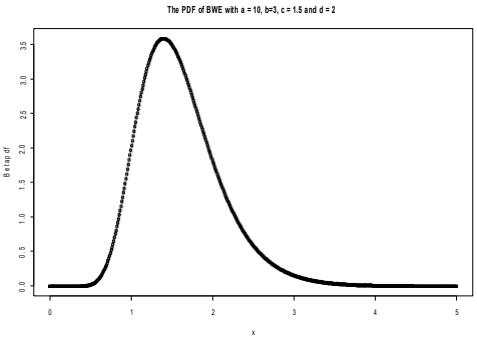


Fig 3. Plot of PDF of BWE at (a,b) = (20,3) Fig 4. Plot of PDF of BWE at (a,b) = (10,2.5)

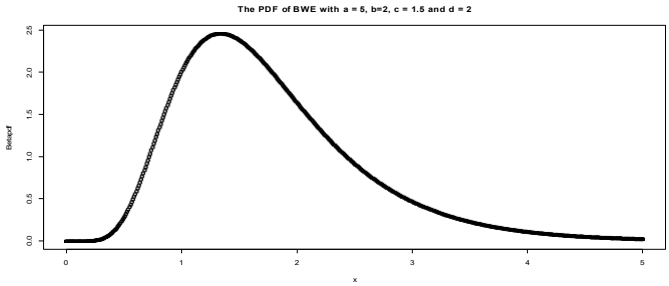


Fig 5 Plot of PDF of BWE at (a,b) = (5,2)

2.3 Cumulative Distribution Function (CDF)

The cumulative distribution function (cdf) of the Beta Weighted Exponential variable Y, derived from the pdf defined in equation (2.2.4) is given as

$$F_{BWE}(y) = \int_0^y f_{BWE}(t) = \frac{1}{B(a,b)} \int_0^y [W(t)]^{a-1}[1 - W(t)]^{b-1} dW(t) \quad (2.3.1)$$

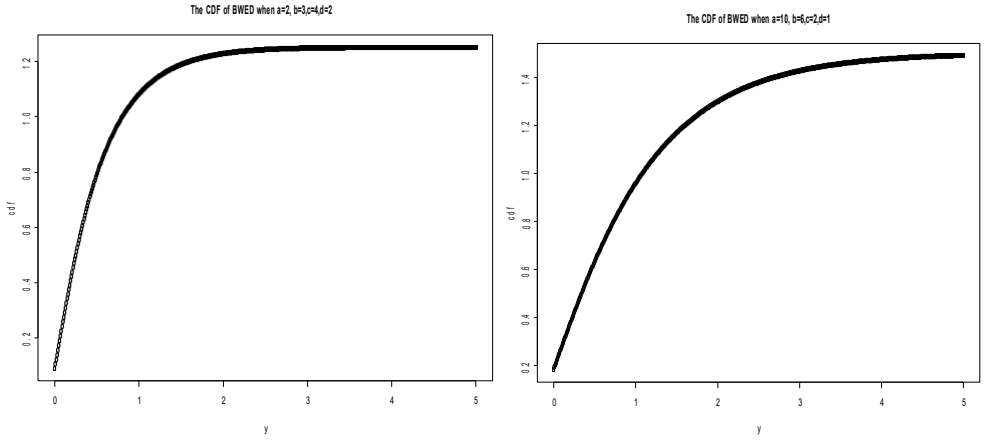
Hence,

$$F_{BWE}(y) = \frac{B(y;a,b)}{B(a,b)} \quad (2.3.2)$$

This is an Incomplete Beta function

Figures 6 and 7 show graphs of cumulative distribution function of the typical Beta weighted exponential distribution for some values of the parameters.

Figures 6 and 7 below are the graph of CDF of BWE Distribution at different values [$a = 2, b = 3, c = \alpha = 4$ and $d = \lambda = 2$; $a = 10, b = 6, c = \alpha = 2$ and $d = \lambda = 1$]



2.4 Survival Rate Function

The survival rate function of a random Beta weighted exponential variable Y with cumulative distribution function F(y) is given by

$$S_{BWE}(y) = 1 - F_{BWE}(y) \quad (2.4.1)$$

From equation (2.3.2), the survival rate function is obtained as

$$S_{BWE}(y) = \frac{B(a,b) - B(y;a,b)}{B(a,b)} \quad .(2.4.2)$$

Figure 8 below is a graph of an example of the survival rate function based on the Beta weighted exponential distribution

2.5 Hazard Rate Function

The hazard rate function of a random Beta weighted exponential variable Y is given as

$$H_{BWE}(y) = \frac{f_{BWE}(y)}{1-F_{BWE}(y)} \quad (2.5.1)$$

Substituting from equations (6) and (8), the hazard rate function simplifies as

$$H_{BWE}(y) = \frac{[W(y)]^{a-1}[1-W(y)]^{b-1}dW(y)}{B(a,b)-B(y;a,b)} \quad (2.5.2)$$

where w(y) is as given in equation (2.2.3)

Figures 8 and 9 are the plots of the survival rate and Hazard Rate Function of BWE Distribution

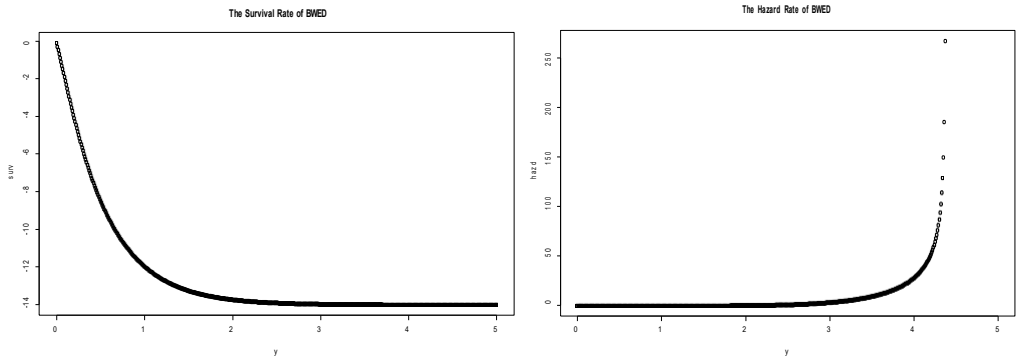


Fig. 8 The Survival Rate Function of BWE Distribution Fig. 9 Hazard Rate Function of BWE Distribution

2.6 The Asymptotic Behaviour

The limits of the PDF are values of expression *(2.2.2)* as $y \to \infty$ and $y \to 0$; it is easy to show that the limits of the probability density function (PDF), $f_{BWE}(y)$, of the Beta weighted exponential variable y are given as $lim_{y\to\infty} f_{BWE}(y) = 0$ and $lim_{y\to 0} f_{BWE}(y) = 0$. This confirms that the Beta weighted exponential distribution has mode(s).

2.7 Special Sub-models

Expression (2.2.2) is important as a generalized model whose sub-models coincide with some extant distributions that correspond respectively to special values of the parameters. Some of the sub-models include the following

i. When, in eq. (2.2.2), $a = b = 1$ and $\alpha = \alpha^{\beta}$ $(\alpha = 0,1 \text{ or } \beta = 1)$, the Beta weighted distribution becomes the weighted Weibull distribution [13]

ii. Putting $a = 1$ in (2.2.2), the generalized models give Lehmann type II exponential weighted distribution

iii. Setting $b = 1$ in expression (2.2.2) reduces the generalized distribution to the exponentiated exponential weighted distribution

Also $a = b = 1$ reduces the generalized distribution to the parent distribution (weighted exponential) distribution.

3 Moments and Generating Function (mgf)

We derived the moment generating function of our generalized Beta Weighted Exponential (BWE) distribution from the works of [11] which was also used in [3] to obtain the moment generating function (MGF) of beta generated distributions. The MGF $M(t) = E(e^{ty})$ was given as

$$M(t) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \int_{-\infty}^{\infty} e^{ty}[F(y)]^{a(i+1)-1} f(y)dy \tag{3.1}$$

Substituting $pdf f(y)$ and $cdf\ F(y)$ as defined below

$$f(y) = \frac{\lambda(1+\alpha)\,[\exp(-\lambda y)\,(1-\exp(-\alpha\lambda y)]}{\alpha}$$

$$F(y) = \frac{[(1+\alpha)(1-\exp(-\lambda y)) + \exp(-\lambda y(1+\alpha))]}{\alpha}$$

into the $MGF\ M(t)$ in equation (3.1) gave

$$M_{BWE}(t) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \int_{-\infty}^{\infty} e^{ty} \left[\frac{[(1+\alpha)(1-\exp(-\lambda y)+\exp(-\lambda y(1+\alpha))]}{\alpha}\right]^{a(i+1)-1} \left[\frac{\lambda(1+\alpha)[\exp(-\lambda y)(1-\exp(-\alpha\lambda y)]}{\alpha}\right] dy \tag{3.2}$$

[13] gave the rth noncentral moment of the class of Weighted Weibull distribution $WW(\alpha.\ \beta;\ \lambda)$ as

$$\mu'_{WWr} = E(Y^r) = \frac{\lambda^{\frac{r}{\beta}}}{\alpha\beta}(1+\alpha\beta)\Gamma\left(\frac{r+\beta}{\beta}\right)\left(1-(1+\alpha\beta)^{\frac{r+\beta}{\beta}}\right) \tag{3.3}$$

which, for $\beta = 1$, like in the case on hand, reduces to

$$\mu'_{WWr} = E(Y^r) = \frac{\lambda^r}{\alpha}(1+\alpha)\Gamma(r+1)[1-(1+\alpha)^{r+1}] \tag{3.4}$$

The rth noncentral moment of the Beta Weighted Weibull distribution would be given as

$$\mu'_{BWE(r)} = \int_0^{\infty} y^r f_{BWE}(y)dy$$

i.e.

$$\mu'_{BWE(r)} = \int_0^{\infty} y^r \left\{\frac{1}{B(a,b)}\right\} [W(y)]^{a-1}[1-W(y)]^{b-1} dW(y)\}$$

where

$$W(y) = \frac{[\eta[1-c(y)])+[c(y)]^{\eta}-1]}{\alpha}, c(y) = e^{-\lambda y}, \ 5 = (1+\alpha)$$

Then,

$$\mu'_{BWE(r)} = [\frac{\lambda^r \eta \Gamma(r+1)(1-\eta^{r+1})}{\alpha B(a,b)}] \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i}.$$

$$\{\int_0^{\infty} [\frac{\{5[1-c(y)]+[c(y)]^{\eta}-1\}^{a(i+1)-1}}{\alpha}] dy\}$$

$$= K[\lambda^r 5\Gamma(r+1)(1-5^{r+1})] \quad (3.5)$$

where $K = \frac{\sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \int_0^{\infty} [[\eta(1-c(y))+c(y)^{\eta}-1]]^{a(i+1)-1} dy}{\alpha B(a,b)}$,

We obtained the first four non-central moments μ'_r by putting $r = 1, 2, 3$ and 4 respectively in eq. 3.5; e.g. μ'_1 is given as

$$\mu'_1 = E_{BWE}(y) = [\frac{\lambda 5\Gamma(2)(1-5^2)}{\alpha B(a,b)}] [\sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i}]$$

Also, central moments μ_r, $r = 1,2,3,4, \ldots \ldots$ are related to noncentral moments μ'_r as

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \mu'_{r-j} \mu'_j, \text{ where } \mu'_1 = \mu \text{ and } \mu'_0 = 1 \quad (3.6)$$

Consequently, the mean and 2nd, 3rd, and 4th moments of the BWE distribution are given as

$$\mu = \mu'_1$$
$$\mu_2 = \mu'_2 - \mu^2$$
$$\mu_3 = \mu'_3 - 3\mu\mu'_2 + 2\mu^3 \text{ and}$$
$$\mu_4 = \mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4$$

where,

$$\mu'_1 = K[\lambda 5(1-5^2)] \quad (3.7)$$

$$\mu'_2 = K[\lambda^2 5\Gamma(3)(1-5^3)] = 2K[\lambda^2 5(1-5^3)] \quad (3.8)$$

$$\mu'_3 = K[\lambda^3 5\Gamma(4)(1-5^4)] = 6K[\lambda^3 5(1-5^4)] \quad (3.9)$$

$$\mu'_4 = K[\lambda^4 5\Gamma(5)(1-5^5)] = 24K[\lambda^4 5(1-5^5)] \quad (3.10)$$

Moments measures of Skewness, γ_1 and of excess kurtosis, γ_2, are respectively given as

$$\gamma_1 = \frac{\mu_3}{\sqrt{\mu_2^3}} \quad (3.11)$$

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 \quad (3.12)$$

Figure 10 below shows plot of Skewness-kurtosis of Survival Time of Guinea Pig Data set

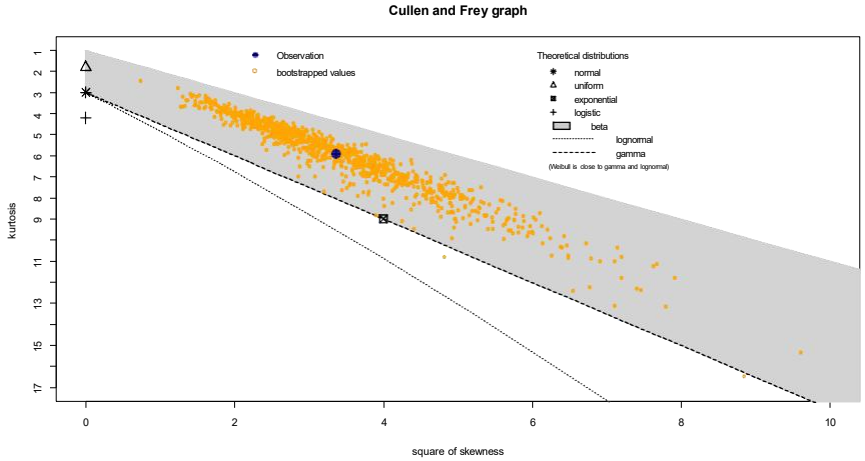


Figure 10: Skewness-kurtosis plot (Survival Time of Guinea Pig Data set)

4 Parameter Estimation

4.1 Maximum Likelihood Estimation (MLEs)

The maximum likelihood estimation (MLEs) of the parameter of *BWE(a, b, λ, α)* distribution has been derived from the study of [6] involving the log-likelihood function and from [3] and [4]; setting $\varphi = (a, b, c, \omega)$, where $\omega = (\lambda, \alpha)$ where ω is a vector of parameters. We had likelihood

$$L_{BWE}(\varphi) = nlogc - nlog[B(a,b)] + \sum_{i=1}^{n} log[f(y;\varphi)] + (a-1)\sum_{i=1}^{n} log\,[F(y;\varphi)] (b - 1) \sum_{i=1}^{n} log\,[1 - F(y;\varphi)] \quad (4.1.1)$$

$$L_{BWE}(\varphi) = Const - nlog[B(a,b)] + \sum_{i=1}^{n} log[f(y;\varphi)] + (a-1)\sum_{i=1}^{n} log\,[F(y;\varphi)](b - 1)i=1nlog\ \ [1-Fy;\varphi] \quad (4.1.2)$$

Taking partial derivative of (4.1.2) with respect to (a, b, λ, α), we get

$$\frac{\partial L_{BWE}(\varphi)}{\partial a} = -nlog(a,b) + (a-1)\sum_{y=1}^{n} log\,[F(y;\varphi] \quad (4.1.3)$$

$$\frac{\partial L_{BWE}(\varphi)}{\partial b} = -nlog(a,b) + (b-1)\sum_{y=1}^{n} log\,[1 - F(y;\varphi] \quad (4.1.4)$$

$$\frac{\partial L_{BWE}(\varphi)}{\partial \lambda} = \sum_{y=1}^{n} log\,[\frac{\frac{6}{6\lambda}[f(y;\varphi)]}{f[y;\varphi)]}] + (a-1)\sum_{y=1}^{n} log\,[\frac{\frac{6}{6\lambda}[F(y;\varphi)]}{F(y;\varphi)}] + (b-1)\sum_{y=1}^{n} log\,[\frac{\frac{6}{6\lambda}[1-F(y;\varphi)]}{1-F(y;\varphi)}] \quad (4.1.5)$$

$$\frac{\partial L_{BWE}(\varphi)}{\partial \alpha} \sum_{y=1}^{n} log\,[\frac{\frac{6}{6\alpha}[f(y;\varphi)]}{f[y;\varphi)]}] + (a-1)\sum_{y=1}^{n} log\,[\frac{\frac{6}{6\alpha}[F(y;\varphi)]}{F(y;\varphi)}] + (b-1)\sum_{y=1}^{n} log\,[\frac{\frac{6}{6\lambda}[1-F(y;\varphi)]}{1-F(y;\varphi)}] \quad (4.1.6)$$

Equations (4.1.3) to (4.1.6) can be solved using iteration method (Newton Raphson) to obtain $a^{\wedge}, b^{\wedge}, \alpha^{\wedge}, \lambda^{\wedge}$ the MLE of (a, b, λ, α) respectively.

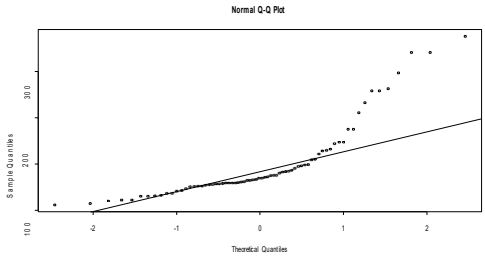
5 Result and Discussion

5.1 Result of Survival Time of Guinea Pig Data

This study used the data by [10] and studied by [5] on the survival times of guinea pigs injected with different amount of tubercle bacilli. The data are given below:

12 15 22 24 24 32 32 33 34 38 38 43 44 48 52 53 54 54 55 56 57 58 58 59 60 60 60 60 61 62 63 65 65 67 68 70 70 72 73 75 76 76 81 83 84 85 87 91 95 96 98 99 109 110 121 127 129 131 143 146 146 175 175 211 233 258 258 263 297 341 341 376.

Figures 11 – 14 shows Normal Quantile Plot, Density Plot, Histogram and Empirical Density and Cumulative Distribution plot of the data.



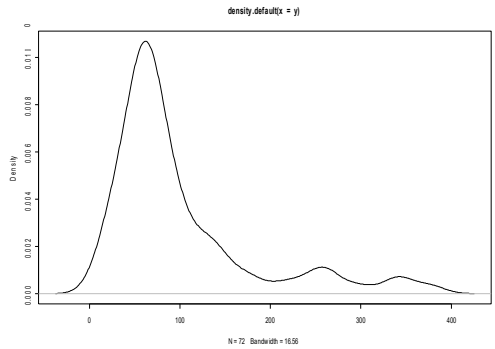


Fig. 11 Normal quantile of survival times of pigs Fig. 12 Density of survival times of pigs

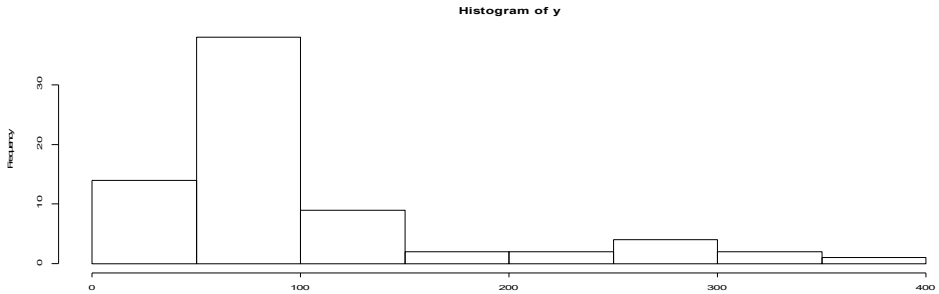


Fig. 13:. Histogram of survival times of pigs

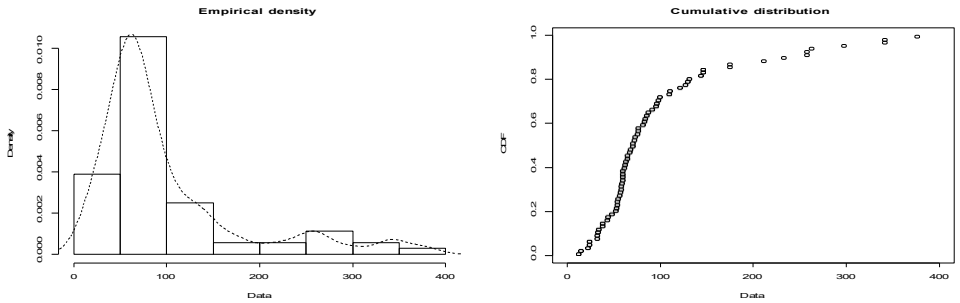


Fig. 14 Histogram and Empirical Density of survival times of pigs

Min	1st Q	Med	Mean	3rd Q	Max	Skewness	Kurtosis
12.00	54.75	70.00	99.82	112.80	376.00	1.7962	5.6144

Table 1: Descriptive Statistics for survival time of Guinea pigs in days.

Model	Estimate, Standard Error in Parentheses and p-value in asterisks						
	$a^$	$b^$	$\lambda^$	$\alpha^$	AIC	BIC	LogLik
BWE	1.000e+00 (5.318e-04) <2e-16 ***	9.619e-01 (7.038e-02) <2e-16 ***	2.127e-02 (1.774e-03) <2e-16 ***	5.210e-04 (5.343e-05) <2e-16 ***	812.839	821.9457	-402.4195
EWE when b=1	1.11031 (5.68100) <2e-16 ***	1	2.12286 (1.01403) < 2e-16 ***	8.59040 (2.96594) 0.00378 **	834.7892	841.6192	-414.3946
LWE at a=1	1	1.11031 (5.68100) <2e-16 ***	2.12286 (1.01403) < 2e-16 ***	8.59040 (2.96594) 0.00378 **	834.7892	841.6192	-414.3946
WE When a= =1	1	1	1.10281 (1.00401) < 2e-16 ***	8.73212 (2.96672) 0.01412**	846.4306	850.9839	-421.2153

Table 2: MLEs of the model parameters, the corresponding Standard Error and p-value

5.2 Discussion

We estimate parameters of the BWE, EWE, LWE and WE respectively using R software codes. Table 1 contains the descriptive statistics of the data set. The skewness value clearly reveals some asymmetry in the empirical distribution while the kurtosis measures the weight of tails in relation to the normal distribution [7]; the result in the Table 1 also shows excess of kurtosis; hence, the data calls for a more robust distribution/model that can accommodate all types of risk function as embedded in BWE distribution. Figure 10 shows the skewness-kurtosis feature of the data set. However, Figures 1 – 5 show the plot of the PDF of BWE distribution for different values of the

parameters; the plots indicate positive and negative skewness i.e. the bathtub shape. Figures 6 and 7 show the plots of CDF of the BWE distribution at given values of the parameters and Figures 8 and 9 are plots of survival and hazard rates respectively. The hazard function is unimodal.

Table 2 above features the estimate of parameters $(a^{\wedge}, b^{\wedge}, \lambda^{\wedge}, \alpha^{\wedge})$ the log-likelihood, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). In a nutshell, the values of AIC and BIC for the BWE distribution model were smaller than for other models; on this score, the BWE model provides a better representation of the Guinea pig data set. The asymptotic covariance matrix of the maximum likelihood estimates for the BWE distribution, which was generated from the inverse of Fisher's information matrix and is given as

$$\Sigma = I \begin{matrix} 2.827714e-07 & -1.020188e-07 & 8.075292e-08 & -7.664858e-06 \\ -1.020188e-07 & 4.953495e-03 & -4.946238e-06 & -4.955421e-06 \\ 8.075292e-08 & -4.946238e-06 & 3.145951e-06 & 1.704223e-07 \\ -7.664858e-06 & -4.955421e-06 & 1.704223e-07 & 2.855199e-09 \end{matrix}]$$

6 Conclusion

The article presented the Beta Weighted Weibull Exponential distribution, a sub-model of which are various existing models, as a generalised model. Moments, moment generating function, skewness, kurtosis, and maximum likelihood estimate are some of the statistical aspects of the suggested distribution that were calculated analytically. Consequently, the PDF, CDF, survival rate, and hazard rate functions were generated using the R programming language. The beta weighted exponential (BWE) distribution has a log-likelihood of -402.4195, while the weighted exponential (WE) distribution has a log-likelihood of -421.2153. AIC and BIC, the selection criteria, for the BWE were 812.839 and 821.9457, respectively, while for the WE they were 846.4306 and 850.9839. It follows that BWE distribution is superior than WE distribution in terms of efficiency. When applied to guinea pig data, the BWE distribution proved to be more flexible and closely matched the distribution of the data, whereas its natural rivals failed to do so.

References

Apinkentete, Famoye, and Lee's [1] The beta Pareto Distribution. Statistical Review, 42(6), 547-563. http://dx.doi.org/10.1080/0233180801983876 (2008).

The normal distribution is one kind of distribution, according to Azzalini [2]. Volume 12, Issue 17, pages 171–178, Scandinavian Journal of Statistics (1985).
[3] Finding the Life Length of Components using the Beta Weighted Weibull Distribution, by Badmus and Bamiduro. Volume 11, Issue 2, pages 91-107, Journal of Statistics: Advances in Theory and Applications (2014).
In their paper "Some Statistical Properties of Exponentiated Weighted Weibull Distribution," Badmus and Bamiduro [4] discuss this topic. A statistical analysis of SOP transactions. Pages

1–11 of Volume 1, Issue 2 (2014).
(Bjerkedal, T., et al., 2005) American guinea pigs inoculated with varying concentrations of tuberculosis-causing bacteria developed resistance. Published in the Journal of Hygiene 72, 130–148 (1960).
Referenced in [6] Cordeiro et al., Generalised Beta Generated distributions. ICMA facility. Financial Discussion Papers DP 2011-05.
[7] A package for fitting distributions in R, developed by Delignette-Muller and Dutang (2014), available at http://cran.r-project.org/package. 1–23.
[8] The beta-weibull distribution was developed by Famoye, F., Lee, C., and Olugbenga, O. The citation comes from the Journal of Statistical Theory and Applications, volume 4, issue 2, pages 121–138. In their 1999 publication "Generalised Exponential Distribution," Gupta and Kundu outlined a method for calculating the expected value of a random variable in a statistical model.
In 2009, Gupta and Kundu published an article in Statistics titled "A new class of weighted exponential distributions" (43(6), 621-634).
Analysis of L-moments and distribution estimate using linear combinations of order statistics [11] by Hosking, J.R.M. A publication from the Royal Statistical Society B, volume 52, pages 105–124, in the year 1990.
The following is a reference to Jones's (2004) work on distribution families: test 13, 1-43. Ramadan, Mahdy, [13]. A class of weighted weibull distributions and their features. Proceedings of the American Mathematical Society, Volume 6, Issue 1, Pages 35–45. 2013 was the year.
The moments of the exponentiated Weibull distribution are discussed in [14] by Nadarajah and Kotz. Scientific Expression. Review of Scientific Methods, 35, 253-256 (2005).